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A Mechanical Model of Brownian Motion in Half-Space

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We consider the motion of a heavy mass in an ideal gas in a semi-infinite system, with elastic collisions at the boundary. The motion is determined by elastic collisions. We prove in the Brownian motion limit the convergence of the position and velocity process of the heavy particle to a diffusion process in which velocity and position remain coupled.

KEY WORDS: Brownian motion; ideal gas; elastic collision.

1. INTRODUCTION

We consider the motion of a massive particle (molecule) in an ideal gas of identical point particles (atoms) in a semi-infinite d-dimensional space ($d \geqslant 2$), the boundary being a wall which elastically reflects all the particles.

We consider the system in the Brownian motion limit: The masses of the atoms tend to zero, while the mass and the size of the molecule are kept constant and the density ρ and the velocities v of the atoms are scaled like $\rho \approx 1/\sqrt{m}$ and $\langle |v| \rangle \approx 1/\sqrt{m}$, with $\langle \cdot \rangle$ denoting the average. The position and velocity of the molecule thus become a family of stochastic processes $(X_m(t), V_m(t), t \geqslant 0)$ on the space of the initial conditions of the ideal gas.

In ref. 1 it is shown that in the infinite system without the presence of the wall, the position–velocity process of the molecule $(X_m(t), V_m(t); t \geqslant 0)$ converges in distribution to an Ornstein–Uhlenbeck process when $m \to 0$. The proof uses a natural Markov approximation of the mechanical motion, based on the fact that only "fast" atoms ($v \approx 1/\sqrt{m}$) have an effect on the

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motion of the molecule, but recollisions with fast atoms do not occur when m tends to zero.

Due to the presence of the wall, it is clear that fast atoms also recollide with the molecule and an entirely different strategy of proof had to be adopted to tackle the problem.

Before entering a more detailed description of this, we note that a fast atom spends, from its first until the last recollision, a time of roughly $\sqrt{m}$ with the molecule. Since by the laws of elastic collisions we have for the change of velocities

$$\Delta V \approx mv, \qquad \Delta v \approx -2v$$

we may expect that the effect of recollisions with a single atom does not spread over a macroscopic time. Given the results of ref. 1, we may therefore expect in the semi-infinite system the convergence of the process $(X_m(t), V_m(t); t \geqslant 0)$ as $m \to 0$ toward a diffusion process in which position and velocity are coupled, since the recollisions depend on the distance of the molecule to the wall. We shall show that this is indeed the case.

The convergence result is proved via relative compactness of the family of measures induced by $(X_m(t), V_m(t); t \geqslant 0)$ on paths space and by identifying a unique limit point by the martingale characterization of diffusion processes.[6] This is a very natural procedure, noting the following. Let $i = 1,..., N(t)$ denote a time-ordered labeling of the different atoms which collide with the molecule until time t. Then we may write

$$V_m(t) - V(0) = \sum_{i=1}^{N(t)} f_m(v_i, x_i, t_i; t)$$

with $f_m(v_i, x_i, t_i; t)$ denoting the total effect on the molecule during $[0, t]$ of atom i, colliding *first* at t_i with precollision velocity v_i and colliding at the place x_i on the surface S of the molecule. (Note that f_m contains the effect of *all* recollisions of atom i.)

Using the notation of point processes, i.e., counting measures, we may write the sum as

$$V_m(t) - V_m(0) = \int \int_0^t N_m(d\tau, dv, dx)\, f_m(x, v, \tau; t) \tag{1.1}$$

with $N_m(d\tau, dv, dx)$ denoting a point process on $\mathbb{R}^d \times S$, which is very roughly speaking like a Poisson process with rate

$$\begin{aligned} R_m &\approx E(N_m(d\tau.\, dv, dx))/V_m(s < \tau)) \\ &\approx \rho_m\, |v - V_m(t)|\, h_m(v)\, dv\, dx\, d\tau \end{aligned} \tag{1.2}$$

$h_m(v)$ is the velocity distribution of the ideal gas, and $E(\cdot/\cdot)$ denotes the conditional expectation. Thus, $N_m - R_m$ is a martingale difference, since by definition its conditional expectation is zero. Hence, we might think of the R_m integral of f_m as representing "the generator L_m of the process V_m, acting on V," i.e., we are close to writing, using (1.1) and (1.2),

$$E\left(V_m(t) - V_m(s) - \int_s^t L_m V_m(u)\, du / V_m(\tau < s) \right) \approx 0 \qquad (1.3)$$

If L_m is close to a generator of a diffusion, we formulate indeed a martingale characterization (very roughly speaking). This is the key observation and leading idea. As usual in stochastic integrals like (1.1), to proceed to (1.3) we need the function $f_m(v, x, \tau; t)$ to be nonanticipating, i.e., not depending on the future of $V_m(t)$, $t > \tau$. But this is of course not our case, due to the recollisions. The work is then to approximate the effect of collisions of one atom by a function depending only on the collision parameters of the atom's first collision, i.e., $f_m(v, x, \tau; t)$ by $\tilde{f}_m(v, x, \tau)$, which is clearly possible by the first observation on the recollision time.

As usual in these proofs, we introduce a stopping time, which helps to control error terms and which may be removed for the limit process. Here we stop the process when the molecule becomes too fast or when the molecule comes too close to the wall. In fact, in the limit process the molecule never reaches the wall, i.e., the drift and the diffusion coefficients become likewise singular at the wall.

The molecule is represented by a cube of fixed orientation with one face parallel to the wall. This choice removes serious complications due to the geometry of more general shapes. One should note, however, that our method makes it possible in principle to handle also the case of two or more molecules of convex shapes, which is the physically most interesting case.

The results are described in detail in the next section. We state them for two and three dimensions. The result in three dimensions is less restrictive on the velocity distribution of the ideal gas. The proof is explicitly written out for the two-dimensional case, mainly for ease of notation and for a better presentation of the probabilistic method involved. In Section 3 we extract the relevant details of the point process. Bounds on the recollisions and tightness of the family of induced measures are discussed in Section 4. The Martingale problem is then considered in Section 5. We close with an Appendix, where we show that the molecule in the limit process never reaches the wall.

2. THE MODEL AND RESULTS

We first introduce the heat bath. Let $\Gamma = (R^{d-1} \times R^+) \times R^d$, $d = 2, 3$, be the one-particle phase space, $B(\Gamma)$ its Borel σ-algebra, on which we define the measure

$$d\nu_m = \rho_m \, dq \, h_m(v) \, dv, \qquad (q, v) \in \Gamma \tag{2.1}$$

with $\rho_m = m^{-1/2}\rho$, $\rho > 0$, velocity distribution density $h_m(v) = m^{d/2} h(m^{1/2} v)$, and $dq\, dv$ the Lebesgue measure on R^{2d}.

We define the Poisson field $(\Omega, F, \mathbf{P}_m)$ built on $(\Gamma, B, d\nu_m)$ as follows: for any $B_1, B_2, ..., B_n$ disjoint sets of $B(\Gamma)$, let

$$N(B_i) = \{\text{the number of particles with coordinates } (q, v) \in B_i\}$$

Then for any $k_1, k_2, ..., k_n$ positive integers

$$\mathbf{P}_m(\{\omega \in \Omega \colon N(B_i) = k_i,\ i = 1, ..., n\}) = \prod_{i=1}^{n} \exp[-\nu_m(B_i)] \frac{\nu_m(B_i)^{k_i}}{k_i!} \tag{2.2}$$

where $\omega = (q_i, w_i)_{i \in Z}$ represents an initial configuration of the bath particles (atoms).

The molecule is taken as a d-dimensional cube of side L and mass M. We place the molecule into the bath, away from the wall and one face parallel to the wall, removing from the initial configuration ω all those particles that are in the closed region to be occupied by the molecule. The Poisson system obtained in this way is again denoted by $(\Omega, F, \mathbf{P}_m)$.

We now define the dynamics. The orientation of the molecule is fixed forever (infinite moment of inertia). The atoms interact with the molecule and the wall by elastic collisions. In between collisions the molecule and the atoms move freely. To describe the collision between an atom with velocity v and the molecule with velocity V, let $v_n = (e_n \cdot v)\, e_n$, $V_n = (e_n \cdot V)\, e_n$, and $v_t = v - v_n$, $V_t = V - V_n$, where e_n denotes the outgoing orthonormal vector of the surface hit by the atom. For the postcollision velocities $v' = (v'_n, v'_t)$ and $V' = (V'_n, V'_t)$ we then have that

$$\begin{aligned} v'_t &= v_t, \qquad V'_t = V_t \\ v'_n &= -(1-\alpha)\, v_n + (2-\alpha)\, V_n, \qquad V'_n = (1-\alpha)\, V_n + \alpha v_n \end{aligned} \tag{2.3}$$

where $\alpha = 2m/(M+m)$.

There are collision situations, however, which cannot be dissolved mechanically. An atom may collide with an edge of the molecule, two or more atoms may collide simultaneously with the molecule, and also infinitely many collisions within a finite amount of time may occur. For given initial value (X_0, V_0) for the molecule, we collect all $\omega \in \Omega$ for which these bad events occur in $\Omega_b(X_0, V_0) \subset \Omega$ and for $\omega \in \Omega_b(X_0, V_0)$ we simply place the trajectory of the molecule in a cementery Δ.

For $\omega \in \Omega \backslash \Omega_b(X_0, V_0)$ the position–velocity process $(X_m, V_m) = (X_m(t), V_m(t); t \geqslant 0)$ of the molecule is now defined as follows: the molecule will move freely according to its initial velocity until the time τ_1 of its first collision with an atom if $\tau_1 < \tau_1^* = \inf\{t \geqslant 0: d(X_0 + V_0 t) = 0\}$, where $d(X)$ denotes the minimal distance of the molecule from the wall, when its center is placed at the point X.

If $\tau_1 \geqslant \tau_1^*$, we shall stop the motion at time $t = \tau_1^*$; i.e., $(V_m(t); t \geqslant 0) = (V_m(t \wedge \tau_1^*); t \geqslant 0)$.

If $\tau_1 < \tau_1^*$, the velocity of the molecule and of the colliding atom will change according to (2.3). Afterward the molecule will move with constant velocity $V_m(\tau_1^+) = V_0'$ until the time τ_2 of the next collision if $\tau_2 < \tau_2^* = \inf\{t > \tau_1 : d(X_m(\tau_1) + V_m(\tau_1^+)(t - \tau_1) = 0\}$, where $X_m(\tau_1) = X_0 + V_0 \tau_1$; otherwise the motion is stopped at time τ_2^*, etc.

Thus, we otain $(V_m(t); t \geqslant 0)$ for all $\omega \in \Omega \backslash \Omega_b(X_0, V_0)$ and $X_m(t) = \int_0^t du\, V_m(u)$.

For $\omega \in \Omega_b(X_0, V_0)$ we set, for all t, $V_m(t) = \Delta$ and $X_m(t) = \Delta$.

One may follow the one-dimensional analysis of Holley[3] to see that in our case $V_m(\cdot, \cdot)$ is a function from $[0, \infty) \times \Omega$ to $R^d \cup \{\Delta\}$ right continuous in t and for any fixed t measurable in ω. Hence, also the process $(X_m(t), V_m(t); t \geqslant 0)$ is measurable in ω for any fixed t.

Following the arguments given in refs. 1 and 2, one can easily show that for almost all (X_0, V_0) and for $m \geqslant 0$

$$\mathbf{P}_m(\Omega_b(X_0, V_0)) = 0 \tag{2.4}$$

provided $h(v)$ has a finite fourth moment.

We may realize the process (X_m, V_m) on the path space $D([0, \infty)) \times D([0, \infty))$ endowed with the Shorohod topology with induced measure $\bar{\mathbf{P}}_m$. Thus, we may consider the weak convergence of the family of processes $(X_m, V_m)_{m \geqslant 0}$ on the path space, i.e., the weak convergence of the measures $\bar{\mathbf{P}}_m$.

In ref. 1, it is shown that the motion of the molecule in the system without wall converges as $m \to 0$ to an Ornstein–Uhlenbeck process. One might therefore, think, that the limit motion now becomes an Ornstein–Uhlenbeck process with reflection at the wall. This is not the case. More careful thought shows that the drift and the diffusion coefficient of the velocity process in the limit will depend on the distance of the molecule from the wall. In fact, we shall show that in the limit the molecule will never reach the wall. When the molecule is near the wall, most collisions on the side facing the wall will be recollisions and thus the statistical effect due to atoms which did not collide before is changed.

Indeed, rough estimates show that the diffusion coefficient may be obtained by computing the effect of recolliding atoms assuming the molecule does not move in between recollisions. Effects due to the change of position and velocity of the molecule enter the drift coefficient only.

In the case of thermal equilibrium, i.e., if the velocity distribution density for the atoms and for the molecule are Maxwellian at the same temperature, one knows by the Einstein relation that the drift is proportional to the diffusion. This provides a heuristic checking of our theorem.

To prove the convergence result, we need stronger conditions on the velocity distribution density $h(v)$. Let $(e_i)_{i=1,\dots,d}$ be an orthonormal basis with e_d the outward normal of the wall; then we assume that

$$\iint v_i \, |v_i| \, h(v) \, dv = 0 \qquad \text{for} \quad i = 1,\dots, d \tag{2.5}$$

$$\iint [|v_d|/\max(|v_i|, i \leqslant d-1)]^{1+\lambda} \, |v_d| \, h(v) \, dv < \infty \qquad \text{for some} \quad \lambda > 0 \tag{2.6}$$

$$\Phi_{k,i} = \iint |v_i|^k \, h(v) \, dv < \infty \qquad \text{for} \quad i \leqslant d \text{ and } k \leqslant \max\{5, (4-3\lambda)/\lambda + 1\} \tag{2.7}$$

Theorem 2.1. Suppose that (2.5)–(2.7) hold. Let $\sigma(X)$ be a diagonal matrix with elements

$$\sigma_{i,i}(X) = (4L\rho\Phi_{3,i}M^{-2})^{1/2} \equiv \sigma_i \qquad \text{for} \quad i = 1,\dots, d-1$$

$$\sigma_{d,d}(X) = \sigma_d \left[1 + (\Phi_{3,d})^{-1} \iint_{v_d > 0} h(v) \, dv \, v_d^3 2F(\eta(X, v)) \right]^{1/2}$$

and let $b(X, V) \in R^d$ with components

$$b_i(X, V) = \, = (4L\rho\Phi_{1,i}M^{-1}) \, V_i \equiv b_i V_i \qquad \text{for} \quad i = 1,\dots, d-1$$

$$b_d(X, V) = b_d \left[1 + \Phi_{1,d}^{-1} \iint_{v_d > 0} dv \, h(v) \frac{\partial}{\partial v_d} v_d^2 F(\eta(X, v)) \right] V_d$$

with

$$\eta(X, v) = \frac{L \, |v_d|}{2d(x) \max(|v_i|; i \leqslant d-1)}$$

$$F(\eta) = \eta^{-1} \int_0^\eta dy \, 2[y] = \eta^{-1} \int_{\eta-1}^\eta dy \, [y][y+1] \qquad \text{for} \quad d = 2$$

$$F(\eta) = \eta^{-2} \int_{\eta-1}^\eta dy \int_{\eta-1}^\eta dz \, [y \wedge z][y \wedge z + 1] \qquad \text{for} \quad d = 3$$

where $[(\cdot)]$ denotes the integer part of $(\cdot)$. Then the stochastic differential equation

$$\begin{aligned} dX(t) &= V(t)\,dt \\ dV(t) &= -b(X(t), V(t))\,dt + \sigma(X(t))\,dW(t) \\ X(0) &= X_0, \qquad V(0) = V_0 \end{aligned} \tag{2.8}$$

with $W(t)$ d-dimensional Brownian motion, defines uniquely a process $(X(t), V(t); t \geqslant 0)$ for all (X_0, V_0), $d(X_0) > 0$, and for almost all (X_0, V_0), $d(X_0) > 0$, the family of processes $(X_m(t), V_m(t); t \geqslant 0)$ converges weakly as $m \to 0$ to $(X(t), V(t); t \geqslant 0)$.

Remarks. 1. Note that in the Maxwellian case, i.e., $h(v) = (2\pi\beta^{-1})^{d/2} \exp(-\beta\,|v|^2/2)$, (2.6) for $d = 3$ holds for any $0 < \lambda < 1$ but it does not hold for $d = 2$.

2. For $d = 3$ and $h(v)$ Maxwellian of parameter β, the process $(V(t), t \geqslant 0)$ has a unique invariant measure

$$\rho(V)\,dV = (2\pi\beta_M^{-1})^{-3/2} \exp(-\beta_M\,|V|^2/2)\,dV$$

with

$$\beta_M = 2b/M\sigma^2 = 2\Phi_{1,i}\Phi_{3,i}^{-1} = \beta \tag{2.9}$$

To check (2.9), we need only show that

$$\Phi_{3,d}^{-1} \iint_{v_d > 0} h(v)\,dv\,v_d^3 2F(\eta(X, v)) = \Phi_{1,d}^{-1} \iint_{v_d > 0} h(v)\,dv\,\frac{\partial}{\partial v_d}\,[v_d^2 F(\eta(X, v))]$$

and this follows easily by integration by parts.

The last argument shows that we need only that $h(v)$ is a product of a Maxwellian in the d direction and an arbitrary function [satisfying (2.5)–(2.7)] for the existence of a stationary Maxwellian distribution with different β_M in the different directions. The theorem also asserts that the limit process is well defined for all times; this means that $d(X(t))$ will never be zero. We prove this in the Appendix.

Hence, for $(f_1, f_2) \in D([0, \infty)) \times D([0, \infty))$ and B, $\delta > 0$ let

$$\tau_{B,\delta}(f_1, f_2) = \inf\{t \geqslant 0\colon d(f_1(t)) < \delta \text{ or } |f_2(t)| \geqslant B\} \tag{2.10}$$

Then we have that for any $t > 0$

$$\lim_{B \to \infty, \delta \to 0} \mathbf{P}(\{\tau_{B,\delta}(X, V) < t\}) = 0 \tag{2.11}$$

where $\mathbf{P}$ is the probability law induced by the process $(X(t), V(t); t \geqslant 0)$. Therefore, by standard arguments (see, for example, Lemma 11.1.1, ref. 6) we need only show the convergence part of Theorem 2.1 for $\bar{\mathbf{P}}_m^{B,\delta}$, the probability law induced by the stopped process $(\bar{X}_m(t), \bar{V}_m(t); t \geqslant 0)$, to $\mathbf{P}^{B,\delta}$, the probability law induced by $(X(t), V(t); t \geqslant 0)$ the stopped limit process, where

$$\begin{aligned} \bar{X}_m(t) &= X_0 + \int_0^t du\, \bar{V}_m(u) \\ \bar{V}_m(t) &= V_m(t \wedge \tau_{B,\delta}(X_m, V_m)) \end{aligned} \tag{2.12}$$

The rest of the paper will be consumed by the proof of the convergence. We shall give the proof of the result only for $d=2$ and assuming that

$$h(v) = 0 \qquad \text{for all} \quad v \text{ s.t. } |v_1| < v, \quad \text{for some } v > 0 \tag{2.7'}$$

so that (2.6) is trivially satisfied. This way we avoid pure technicalities and heavy notations due to higher dimensionality.

In the next section we analyze the collision process in more detail.

3. PRELIMINARIES

Let $\partial = \bar{\partial} \cup \partial_1$ be the boundaries of the molecule, placed at the origin, with ∂_1 denoting the side facing the wall, which we assume to coincide with the x axis. For $\Lambda = \partial \times R^2$ let $B(\Lambda)$ be the σ-algebra of the Borel sets of Λ. For given $\omega \in \Omega \backslash \Omega_b(X_0, V_0)$ we define collision times

$$\tau(q_i, w_i) = \inf\{t \geqslant 0\colon [X_m(t) - (q_i + w_i t)] \in \partial\} \tag{3.1}$$

and collision points

$$x\{\tau(q_i, w_i)\} = X_m(\tau(q_i, w_i)) - (q_i + w_i \tau(q_i, w_i)) \tag{3.2}$$

and let $\{t_j\}_{j \in N}$, $t_0 = 0$, denotes the natural ordering of the collision times. If $t_j = \tau(q_i, w_i)$, we denote now by (x_j, v_j) the pair $(x\{t_j\}, w_i\}$. Finally, we set, for any B, $\delta > 0$,

$$\tau_{B,\delta} \equiv \tau_{B,\delta}(X_m, V_m) = \inf\{t \geqslant 0\colon d(X_m(t)) < \delta \text{ or } |V_m(t)| \geqslant B\} \tag{3.3}$$

For any $t > 0$ and $A \in B(\Lambda)$ we define the point process

$$N(t, A) = \sum_{j \geqslant 1} \chi(t_j < t \wedge \tau_{B,\delta})\, \chi((x_j, v_j) \in A) \equiv \int_0^t \int_A dN(\tau, x, v) \tag{3.4}$$

Defining $G_t = \sigma(N(s, A); 0 \leqslant s \leqslant t, A \in B(\Lambda))$, the process $(N(t, A), t \geqslant 0)$ can be decomposed as $N(t, A) = M(t, A) + A(t, A)$, where $M(t, A)$ is a $(\mathbf{P}_m, G_t)$-martingale and $A(t, A)$ is a predictable increasing process.[9] We shall show below that

$$A(t, A) = \int_0^t du\, \lambda(u, A), \qquad \lambda(u, A) = 0 \qquad \text{for} \quad u > \tau_{B,\delta} \tag{3.5}$$

where $\lambda(u, A)$ roughly represents the collision rate, which depends of course on the process $(\bar{X}_m(t), \bar{V}_m(t); t \geqslant 0)$.

Due to the presence of the wall, which leads to recollisions, we are not able to compute the collision rate explicitly. But we have that for any $A = A_1 \times A_2 \subset \{(x, v) \in \Lambda : |v_y| > 2B\}$ and $0 \leqslant u \leqslant \tau_{B,\delta}$

$$\lambda(u, (A_1 \cap \tilde{\partial}) \times A_2) = \iint_{A_2} dv\, h_m(v)\, \rho_m \int_{x \in \tilde{\partial} \cap A_1} dx\, |(v - \bar{V}_m(u))_{n(x)}| \tag{3.6}$$

$$\begin{aligned}\lambda(u, (A_1 \cap \partial_1) \times A_2) \geqslant \iint_{A_2} dv\, h_m(v)\, \rho_m \int_{x \in \partial_1 \cap A_1} dx\, |(v - \bar{V}_m(u))_{n(x)}| \\ \times \chi(x \leqslant (2d(\bar{X}_m(u))\, \mathrm{tg}\, v^-) \wedge L)\end{aligned} \tag{3.7}$$

and setting $\sigma_m = L(vm^{-1/2} - B)^{-1}$, $\tau_y(u, v) = 2d(\bar{X}_m(u))(|v_y| + B)^{-1}$, we have

$$\begin{aligned}&\lambda(u, (A_1 \cap \partial_1) \times A_2) \\ &\quad \leqslant \chi(u \geqslant \sigma_m) \iint_{A_2} dv\, h_m(v)\, \rho_m \int_{x \in \partial_1 \cap A_1} dx\, |(v - \bar{V}_m(u))_{n(x)}| \\ &\quad \times \chi(x \leqslant 2d(\bar{X}_m(u))\, \mathrm{tg}\, v^+ \wedge L) \\ &\quad + \chi(u \leqslant \sigma_m) \iint_{A_2} dv\, h_m(v)\, \rho_m \Big\{ \chi(u \leqslant \tau_y(u, v)) \iint_{x \in \partial_1 \cap A_1} dx \\ &\quad + \chi(u \leqslant \tau_y(u, v)) \int_{x \in \partial_1 \cap A_1} dx\, |(v - \bar{V}_m(u))_{n(x)}| \\ &\quad \times \chi(x \leqslant 2d(\bar{X}_m(u))\, \mathrm{tg}\, v^+ \wedge L) \Big\}\end{aligned} \tag{3.8}$$

where $(v - \bar{V}_m(u))_{n(x)} = (v - \bar{V}_m(u))\, e_{n(x)}$, with $e_{n(x)}$ the outgoing orthonormal vector of ∂ at the point x, and

$$\mathrm{tg}\, v^{\pm} = (|v_x| \pm B)(|v_y| - B)^{-1}$$

We remark that for $u \geqslant \sigma_m$ all the atoms which at time zero were in the region between the molecule and the wall are no longer there at time u.

Proof of (3.5). For any $A \in B(\Lambda)$ define $N^n(t, A) = N(t, A) \wedge n = N(t \wedge t_n, A)$; we shall prove that for any $n \geqslant 1$ and $m \geqslant 0$

$$\Lambda^n(t, A) = \int_0^{t \wedge t_n \wedge \tau_{B,\delta}} du\, \lambda(u, A)$$

where $\Lambda^n(t, A)$ is the predictable increasing process associated with $N^n(t, A)$. Hence, (3.5) will follow upon taking the limit $n \to \infty$. For any $0 \leqslant s \leqslant t$ and $\varepsilon > 0$ let $\{u_j\}_{j=1,k}$ be a sequence of times, $u_0 = s$, $u_k = t$, $\sup\{u_{j+1} - u_j; j \geqslant 0\} = \varepsilon$.

Setting $\Delta^n N(j, A) = N^n(u_j, A) - N^n(u_{j-1}, A)$, we have that

$$\begin{aligned}
&\mathbf{E}_m(N^n(t, A) - N^n(s, A) \mid G_s) \\
&\quad = \mathbf{E}_m((N^n(t, A) - N^n(s, A))\, \chi(\forall j\colon \Delta N^n(j, A) = 1) \mid G_s) \\
&\qquad + \mathbf{E}_m((N^n(t, A) - N^n(s, A))\, \chi(\exists j\colon \Delta N^n(j, A) > 1) \mid G_s) \qquad (3.9)
\end{aligned}$$

where $\mathbf{E}_m$ denotes the expectation with respect to $\mathbf{P}_m$.

Now, the second term on the rhs of (3.9) tends to zero as $\varepsilon \to 0$ because it is bounded by $n\mathbf{P}_m(\{\exists j\colon \Delta N^n(j, A) > 1\} \mid G_s)$, which tends to zero as $\varepsilon \to 0$, since the probability of having a multiple collision during $[s, T]$ is zero.

For the first term on the rhs of (3.9) let us introduce for $j \geqslant 1$ the following event:

$$\begin{aligned}
C_j = \{&\text{during } (u_{j-1}, u_j] \text{ the molecule does not collide with particles which} \\
&\text{collided first during } (0, u_{j-1}]\} \cap \{\Delta N^n(j, A) = 1\}
\end{aligned}$$

Note that $\mathbf{P}_m(\{\exists j\colon \Delta N^n(j, A) = 1\} \cap C_j^c \mid G_s)$ converges to zero as $\varepsilon \to 0$, because we may again reduce it to the probability of having a multiple collision during $[s, T]$.

Furthermore, setting, for $j \geqslant 1$,

$$D(u_{j-1}) = \{(q, w) \in \Gamma\colon \tau(q, w) \leqslant u_{j-1} \wedge t_n \wedge \tau_{B,\delta}\}$$

and

$$\begin{aligned}
\tilde{D}(u_{j-1}, A) = \{(q, w) \in \Gamma\colon &\inf\{s > u_{j-1} \wedge t_n\colon \\
&(\bar{X}_m(u_{j-1}) + \bar{V}_m(u_{j-1})(s - u_{j-1}) - (q + ws)) \in A_1\} \\
&\leqslant u_j \wedge t_n \wedge \tau_{B,\delta},\, w \in A_2\}
\end{aligned}$$

we have that

$$\{\Delta N^n(j, A)=1\} \cap C_j=\{N(\tilde{D}(u_{j-1}, A)\backslash D(u_{j-1}))=1\}$$

and hence by the strong Markov property of the Poisson field representing the bath we have that

$$\begin{aligned} &\mathbf{P}_m(\{\Delta N^n(j, A)=1\} \cap C_j \mid G_s) \\ &\quad =\mathbf{E}_m(\nu_m(\tilde{D}(u_{j-1}, A)\backslash D(u_{j-1})) \mid G_s)+o(\varepsilon) \end{aligned} \tag{3.10}$$

Since

$$\begin{aligned} &\mathbf{E}_m(\nu_m(\tilde{D}(u_{j-1}, A)\backslash D(u_{j-1})) \mid G_s) \\ &\quad \leqslant \mathbf{E}_m(\mathbf{E}_m(\nu_m(\{(q, w) \in \Gamma\colon w \in A_2, \\ &\qquad q=x+(w-\bar{V}_m(u_{j-1}))-2(w-\bar{V}_m(u_{j-1}))_{n(x)}\, e_{n(x)})u \text{ for some } x \in A_1 \\ &\qquad \text{and } 0<u \leqslant u_j \wedge t_n \wedge \tau_{B,\delta}-u_{j-1} \wedge t_n \wedge \tau_{B,\delta}\}) \mid G_{u_{j-1}}) \mid G_s) \end{aligned}$$

we obtain from (3.10) that

$$\begin{aligned} &\lim_{\varepsilon \to 0} \mathbf{E}_m\left([N^n(t, A)-N^n(s, A)]\chi\left(\prod_{j=1}^{k}\{\Delta N^n(j, A)=1\} \cap C_j\right)\middle| G_s\right) \\ &\quad =\lim_{\varepsilon \to 0} \sum_{j=1}^{k} \mathbf{P}_m(\{\Delta N^n(j, A)=1\} \cap C_j \mid G_s) \\ &\quad \leqslant \lim_{\varepsilon \to 0} \sum_{j=1}^{k} \mathbf{E}_m\left(\iint_{A_2} dv\, h_m(v)\, \rho_m \int_{x \in A_1} dx\, |(v-V_m(u_{j-1}))_{n(x)}|\right. \\ &\qquad \left.\times(u_{j-1} \wedge t_n \wedge \tau_{B,\delta}-u_{j-1} \wedge t_n \wedge \tau_{B,\delta})\middle| G_s\right) \\ &\quad =\mathbf{E}_m\left(\int_{s \wedge t_n \wedge \tau_{B,\delta}}^{t \wedge t_n \wedge \tau_{B,\delta}} du \iint_{A_2} dv\, h_m(v)\, \rho_m \int_{x \in A_1} dx\, |(v-V_m(u))_{n(x)}| \middle| G_s\right) \end{aligned} \tag{3.11}$$

and (3.11) implies (3.5).

To prove (3.6)–(3.8), we first observe that if $A_1 \cap \partial_1=\varnothing$, the equal sign holds in (3.11) [thus yielding (3.6)]; hence, we need only consider collisions in ∂_1. For those, note that if an atom with velocity $v=(v_x, v_y)$ collides at time $\tilde{u}$, then, if it collided before with the wall, it did so at time $\tilde{u}-d(\bar{X}_m(\tilde{u}))/|v_y|$.

Moreover, if τ^* denotes the time [smaller than $\tilde{u}-d(\bar{X}_m(\tilde{u}))/|v_y|$] at which the molecule and the atom were at the same distance from the wall, then

$$\int_{\tau^*}^{\tilde{u}} du\, |(v - V_m(u))_x| \geqslant x \tag{3.12}$$

and

$$\int_{\tau^*}^{\tilde{u}} du\, V_m(u)_y = 2d(X_m(\tilde{u})) - v_y(\tilde{u} - \tau^*) \tag{3.13}$$

Using this in the rhs of (3.10) and conditioning as before yields (3.7) and (3.8) for $u \geqslant \sigma_m$.

To prove (3.8) for $u < \sigma_m$, note that if $\tau_y(u, v) \geqslant u$, then the colliding atom was at time zero in the region $0 \leqslant y \leqslant d(\bar{X}_m(0))$; thus, the collision may happen at any point $x \in \partial_1$. Therefore, (3.8) will follow again from (3.10)–(3.12).

Finally, we note that for any $u \geqslant 0$ one may obtain from (3.11) the following "basic" upper bound:

$$\lambda(u, A) \leqslant \iint_{A_2} dv\, h_m(v)\, \rho_m \int_{x \in A_1} dx\, |(v - \bar{V}_m(u^-))_{n(x)}| \tag{3.14}$$

Notations. For any function H: $[0, \infty) \times \Lambda \to R^d$ and $A \in B(\Lambda)$ we shall denote

$$\int_0^t \int_A dN(\tau, x, v)\, H(\tau, x, v) = \sum_{j \geqslant 1} H(t_j, x_j, v_j)\, \chi(t_j \leqslant t, (x_j, v_j) \in A) \tag{3.15}$$

If

$$\mathbf{E}_m \left(\int_0^t du \int_A \lambda(u, dx, dv)\, |H(u, x, v)| \right) < \infty$$

where $\lambda(u, dx, dv)$ denotes the random measure $\int_A \lambda(u, dx, dv) = \lambda(u, A)$, we define

$$M_H(t, A) \equiv \int_0^t \int_A dN(\tau, x, v)\, H(\tau, x, v) - \int_0^t du \int_A \lambda(u, dx, dv)\, H(u, x, y) \tag{3.16}$$

and if $(H(t, x, v); t \geqslant 0)$ is a G_t-predictable process, then $(M_H(t, A); t \geqslant 0)$ is a $(\mathbf{P}_m, G_t)$-martingale.[5]

4. TIGHTNESS

We provide first bounds for the number of recollisions that a given atom may have with the molecule. Let $\{t_{j,k}\}_{k\geq 0}$, $t_{j,0}=t_j$, $t_{j,k}<t_{j,k+1}$, be the sequence of the recollision times of the jth atom (i.e., of the atom that collided first at time t_j) and for any $t>0$ let

$$k(t_j, t)=\begin{cases}\sup\{k\geq 0\colon t_{j,k}\leq t\} & \text{if} \quad t_j\leq t\\ 0 & \text{if} \quad t_j>t\end{cases} \tag{4.1}$$

Furthermore, $v_{j,k}^-$ ($v_{j,k}^+$) denotes the pre- (post-) collision velocity of the jth atom at time $t_{j,k}$ and $d(t_{j,k})=d(\bar{X}_m(t_{j,k}))$.

For any $u<u'$, we set

$$R^{1(2)}(u, u')=\{j\colon u\wedge\tau_{B,\delta}\leq t_j<u'\wedge\tau_{B,\delta} \text{ and } |(v^-)_y|>(\leq)\, 2B\} \tag{4.2}$$

Lemma 4.1. For any $0<t<\tau_{B,\delta}$, $j\in R^1(0, t)$, and $x_j\in\partial_1$

$$k^-(t_j, t)\leq k(t_j, t)\leq k^+(t_j, t)$$

where

$$k^-(t_j, t)=\left[\frac{(v_j^-)_y}{2d(t_j)}\left\{\frac{L-x_j}{|(v_j^-)_x|+B}\wedge(t-t_j)\right\}\left(1-\frac{C_1}{|(v_j^-)_y|}\right)\right]$$

$$k^+(t_j, t)=\left[\frac{(v_j^-)_y}{2d(t_j)}\left\{\frac{L-x_j}{|(v_j^-)_x|-B}\wedge(t-t_j)\right\}\left(1+\frac{C_2}{|(v_j^-)_y|}\right)\right]$$

with C_1, C_2 positive constants.

Proof. Set $s_k=t_{j,k}-t_{j,k-1}$. We have that

$$\sum_{k=1}^{k(t_j,t)} s_k\leq(t-t_j)\wedge\frac{L-x_j}{|(v_j)_x|-B} \tag{4.3}$$

and if $(t-t_j)>(L-x_j)/(|(v_j)_x|+B)$, then

$$\sum_{k=1}^{k(t_j,t)} s_k\geq\frac{L-x_j}{|(v_j)_x|+B} \tag{4.3'}$$

and

$$d(t_{j,k-1})-Bs_k\leq d(t_{j,k})\leq d(t_{j,k+1})+Bs_k \tag{4.4}$$

Let, for the moment, $v^k=(v_{j,k})^+$ and $d_k=d(t_{j,k})$ and set $\tilde{s}_k=s_k-d_{k-1}(v^{k-1})^{-1}$ [note that $d_{k-1}(v^{k-1})^{-1}$ is the time needed by the jth atom to reach the wall after the $(k-1)$th recollision]. Then

$$\int_0^{s_k} du\, \bar{V}_m(t_{j,k-1}+u)_y=-d_{k-1}-(v^{k-1})_y\,\tilde{s}_k$$

and therefore

$$2d_{k-1}[-(v^{k-1})_y+B]^{-1}\leqslant s_k\leqslant 2d_{k-1}[-(v^{k-1})_y-B]^{-1} \tag{4.5}$$

and by (4.4)

$$\frac{(v^{k-1})_y+3B}{(v^{k-1})_y+B}d_{k-1}\leqslant d_k\leqslant\frac{(v^{k-1})_y-B}{(v^{k-1})_y+B}d_{k-1} \tag{4.6}$$

Denoting $a_n=-2B/[(v^n)_y+B]^{-1}$ for $n\geqslant 1$ and $a_0=-2B/[(v_j^-)_y+B]$, by (4.5) and (4.6), then it follows easily that

$$\begin{aligned}\sum_{k=1}^{k(t_j,t)}s_k&\leqslant\frac{2d(t_j)}{(v_j^-)_y-B}+\left[\sum_{k=2}^{k(t_j,t)}\frac{2d(t_j)}{2B}a_{k-1}\prod_{n=0}^{k-2}(1+a_n)\right]\chi(k(t_j,t)\geqslant 2)\\ \sum_{k=1}^{k(t_j,t)}s_k&\geqslant\frac{2d(t_j)}{(v_j^-)_y+B}+\left[\sum_{k=2}^{k(t_j,t)}\frac{2d(t_j)}{-(v^{k-1})_y+B}\prod_{n=0}^{k-2}(1+a_n)\right]\chi(k(t_j,t)\geqslant 2)\end{aligned} \tag{4.7}$$

By (2.3) it is easy to check that

$$\begin{aligned}(v_j^-)_y-[\alpha(v_j^-)_y+2B]k&\leqslant(1-\alpha)^k(v_j^-)_y-2kB\\ &\leqslant-(v^k)_y\leqslant(1-\alpha)^k(v_j^-)_y+2Bk\end{aligned} \tag{4.8}$$

For the upper bound note that (4.7) yields

$$\sum_{k=1}^{k(t_j,t)}s_k\geqslant\sum_{k=0}^{k(t_j,t)-1}\frac{2d(t_j)}{-(v^k)_y+B}+\left[\sum_{k=1}^{k(t_j,t)-1}\frac{2d(t_j)}{-(v^k)_y+B}\sum_{j=0}^{k-1}a_j\right]\chi(k(t_j,t)\geqslant 2)$$

By (4.8) we have that

$$\begin{aligned}\sum_{k=0}^{k(t_j,t)-1}\frac{2d(t_j)}{-(v^k)_y+B}&\geqslant\sum_{k=0}^{k(t_j,t)-1}\frac{2d(t_j)}{(v_j^-)_y+(2k+1)B}\\ &\geqslant\frac{2d(t_j)}{(v_j^-)_y+B}k(t_j,t)\left\{1-\frac{B}{(v_j^-)_y+B}[k(t_j,t)-1]\right\}\end{aligned} \tag{4.9}$$

and for $k(t_j,t)\geqslant 2$,

$$\begin{aligned}\sum_{k=1}^{k(t_j,t)-1}\frac{2d(t_j)}{-(v^k)_y+B}\sum_{j=0}^{k-1}a_j&\geqslant\frac{2B}{[(v_j^-)_y+B]^2}\sum_{k=1}^{k(t_j,t)-1}2d(t_j)k\\ &=\frac{2Bd(t_j)}{[(v_j^-)_y+B]^2}[k^2(t_j,t)-k(t_j,t)]\end{aligned} \tag{4.10}$$

From (4.9) and (4.10) we obtain that

$$\sum_{k=0}^{k(t_j,t)} s_k \geqslant \frac{2d(t_j)}{(v_j^-)_y+B} k(t_j,t)\left[1-\frac{2B}{(v_j^-)_y-B} k(t_j,t)\right] \tag{4.11}$$

Then the upper bound will easily follow from the right inequality of (4.3) and (4.11).

Next we shall prove the lower bound. By (4.8) and the upper bound for $k(t_j, t)$ we have that $a_k > 0$ and

$$\sum_{k=0}^{k(t_j,t)-1} a_k \leqslant \frac{-2B}{\alpha(v_j^-)_y+2B} \lg\left[1-\frac{\alpha(v_j^-)_y+2B}{(v_j^-)_y-B} k(t_j,t)\right]$$

Hence

$$\begin{aligned}\sum_{k=1}^{k(t_j,t)} s_k &\leqslant \frac{2d(t_j)}{(v_j^-)_y-B}+\frac{2d(t_j)}{2B}\left[\prod_{j=0}^{k(t_j,t)-1}(1+a_j)-(1+a_0)\right]\\ &=\frac{2d(t_j)}{2B}\left[\prod_{j=0}^{k(t_j,t)-1}(1+a_j)-1\right]\\ &\leqslant \frac{d(t_j)}{B}\left[\exp\left(\sum_{j=0}^{k(t_j,t)-1} a_j\right)-1\right]\\ &\leqslant \frac{d(t_j)}{B}\left\{\left[1-\frac{\alpha(v_j^-)_y+2B}{(v_j^-)_y-B} k(t_j,t)\right]^{-2B/[\alpha(v_j^-)_y+2B]}-1\right\}\end{aligned} \tag{4.12}$$

Setting

$$a=\frac{\alpha(v_j^-)_y+2B}{(v_j^-)_y-B} \qquad \text{and} \qquad c=\frac{\alpha(v_j^-)_y}{\alpha(v_j^-)_y+2B}$$

then $a>0$ and $(1-ak)^{-(1-c)}-1 \leqslant (1-c)\,ak(1-ak)^{-(2-c)}$; therefore, from (4.12) we obtain that

$$\sum_{k=1}^{k(t_j,t)} s_k \leqslant \frac{2d(t_j)}{(v_j^-)_y-B} k(t_j,t)[1-ak(t_j,t)]^{-(2-c)} \tag{4.13}$$

Let k_0 be the root of $y(x)=0$, where

$$y(x)=x-b(1-ax)^{2-c}, \qquad b=\frac{(v_j^-)_y-B}{2d(t_j)}\left[(t-t_j)\wedge\frac{L-x_j}{|(v_j^-)_x|-B}\right] \tag{4.14}$$

Then by the inequalities (4.3), (4.3′), and (4.13) we obtain that $k(t_j, t) \geqslant k_0$.

Theorem 4.1. The family of processes $(\bar{V}_m(t); t \geqslant 0)$ is tight.

Proof. To simplify the notation, we set $V^{\pm}_{j,k} = \bar{V}_m(t^{\pm}_{j,k})$ for all $k \geqslant 1$ and $V^{\pm}_{j,0} = \bar{V}_m(t^{\pm}_j)$. Then for all s, t $(0 < s < t)$, we can write

$$\bar{V}_m(t) = \sum_{j \in R^1(0,t)} \sum_{k=0}^{k(t_j,t)} (V^+_{j,k} - V^-_{j,k}) + \sum_{j \in R^2(0,t)} \sum_{k=0}^{k(t_j,t)} (V^+_{j,k} - V^-_{j,k}) \tag{4.15}$$

By the collision law (2.3) we have that for all $0 \leqslant k \leqslant k(t_j, t \wedge \tau_{B,\delta})$

$$\begin{aligned} V^+_{j,k} - V^-_{j,k} &= \alpha(v^-_{j,k} - V^-_{j,k}) \\ &= \alpha(v^-_j)_n - \alpha(V^-_{j,k})_n - \alpha(2-\alpha) \\ &\quad \times \sum_{i=0,k\geqslant 1}^{k-1} (1-\alpha)^j (V^-_{j,k-i-1})_n + \alpha[1-(1-\alpha)^k](v^-_j)_n \end{aligned} \tag{4.16}$$

and we set

$$\bar{V}_m(t) = y_1(t) + y_2(t) + y_3(t) \tag{4.17}$$

where

$$\begin{aligned} y_1(t) &\equiv \sum_{j \in R^1(0,t)} \alpha[k(t_j, t) + 1](v^-_j)_n \\ y_2(t) &\equiv -\sum_{j \in R^1(0,t)} \sum_{k=0}^{k(t_j,t)} \left[\alpha(V^-_{j,k})_n + \alpha(2-\alpha) \sum_{i=0,k\geqslant 1}^{k-1} (1-\alpha)^i (V^-_{j,k-i-1})_n \right] \\ y_3(t) &\equiv \sum_{j \in R^1(0,t)} [(1-\alpha)^{k(t_j,t)} - 1 + \alpha k(t_j, t)](v^-_j)_n + \sum_{j \in R^2(0,t)} \sum_{k=0}^{k(t_j,t)} (V^+_{j,k} - V^-_{j,k})_n \end{aligned} \tag{4.18}$$

Then the tightness of the process $(\bar{V}_m(t); t > 0)$ follows once we show that the following hold (see ref. 4, Proposition 5.7):

(i) For each n, $n = 1, 2$, there exist β, γ, $c > 0$ and $\delta_m \to 0$ such that for all $|t-s| > \delta_m$, $s < t \leqslant T < +\infty$,

$$\bar{\mathbf{E}}^{B,\delta}_m(|y_n(t) - y_n(s)|^\beta) \leqslant c\, |t-s|^{1+\gamma}$$

(ii) For all $\varepsilon > 0$ and $T < +\infty$

$$\lim_{m \to 0} \bar{\mathbf{P}}^{B,\delta}_m(\{ \sup_{\substack{0 < s < t \leqslant T \\ |t-s| < \delta_m}} |[y_1(t) - y_1(s)] + [y_2(t) - y_2(s)]| > \varepsilon \}) = 0$$

(iii) For all ε and $T < +\infty$

$$\lim_{m \to 0} \bar{\mathbf{P}}^{B,\delta}_m(\{ \sup_{0 \leqslant t \leqslant T} |y_3(t)| > \varepsilon \}) = 0$$

where $\bar{\mathbf{E}}^{B,\delta}_m$ denotes the expectation with respect to $\bar{\mathbf{P}}^{B,\delta}_m$.

Proof of (i). From now on we shall write $\bar{\mathbf{P}}_m$, $\bar{\mathbf{E}}_m$ instead of $\bar{\mathbf{P}}_m^{B,\delta}$, $\bar{\mathbf{E}}_m^{B,\delta}$, respectively. First we shall prove (i) for $(y_2(t); t \geq 0)$ with $\beta = 2$, $\delta_m = m^\theta$, $1 > \theta > 1/2$, and $\gamma = \theta^{-1} - 1$. Noting that $k(t_j, t) = k(t_j, s)$ for $t_j < s$ and $|t_j - s| > \sigma_m$ $[\sigma_m = L(vm^{-1/2} - B)^{-1}]$ and observing the bound on V_m, we obtain that

$$\begin{aligned} &\mathbf{E}_m(|y_2(t) - y_2(s)|^2) \\ &\quad\leqslant \mathbf{E}_m\left(\left[\int_s^t \int_{R^1} dN(\tau, x, v)\, \alpha(k(\tau, t) + 1)^2 B\right]^2\right) \\ &\quad\quad + \mathbf{E}_m\left(\left[\int_{(s-\sigma_m)\wedge 0} \int_{R^1} dN(\tau, x, v)\, \alpha(k(\tau, t) + 1)^2 B\right]^2\right) \end{aligned} \tag{4.19}$$

where we used the point measure $dN(\tau, x, v)$ to express the sum in (4.18) [cf. (3.4) and (3.14)].

The upper bound for $k^+(\tau, t)$ given in Lemma 4.1 and the decomposition (3.16) yield

$$\begin{aligned} &\mathbf{E}_m\left(\left[\int_s^t \int_{R^1} dN(\tau, x, v)\, \alpha(k(\tau, t) + 1)^2 B\right]^2\right) \\ &\quad + 2\mathbf{E}_m\left(\left[\int_{\bar{s}\wedge\tau_{B,\delta}}^{\bar{t}\wedge\tau_{B,\delta}} du \int \lambda(u, dx, dv)\, \alpha(k^+(u, t) + 1)^2 B\right]^2\right) \\ &\quad + 2\mathbf{E}_m\left(\left\{\int_s^t \int_{R^1} [dN(\tau, x, v) - d\tau\, \lambda(\tau, dx, dv)]\, \alpha(k^+(\tau, t) + 1)^2 B\right\}^2\right) \end{aligned} \tag{4.20}$$

Using the basic estimate (3.14), it is easy to show that

$$\mathbf{E}_m\left(\left[\int_{\bar{s}\wedge\tau_{B,\delta}}^{\bar{t}\wedge\tau_{B,\delta}} du \int \lambda(u, dx, dv)\, \alpha(k^+(u, t) + 1)^2 B\right]^2\right) \leqslant C_3(t - s)^2 \tag{4.21}$$

For the second term on the rhs of (4.20) we observe that $k^+(\tau, t)$ is a predictable function and therefore we can use the quadratic variation to estimate as follows:

$$\begin{aligned} &\mathbf{E}_m\left(\left\{\int_s^t \int_{R^1} [dN(\tau, x, v) - d\tau\, \lambda(\tau, dx, dv)]\, \alpha(k^+(\tau, t) + 1)^2 B\right\}^2\right) \\ &\quad = \mathbf{E}_m\left(\int_{\bar{s}\wedge\tau_{B,\delta}}^{\bar{t}\wedge\tau_{B,\delta}} du \int \lambda(u, dx, dv)[\alpha(k^+(u, t) + 1)^2 B]^2\right) \\ &\quad \leqslant C_4 m(t - s) \leqslant C_4(t - s)^2 \end{aligned} \tag{4.22}$$

Thus, from (4.20)–(4.22) we obtain

$$\mathbf{E}_m\left(\left[\int_s^t \int_{R^1} dN(\tau, x, v)\, \alpha(k(\tau, t)+1)^2 B\right]^2\right) \leqslant C_5(t-s)^2 \qquad (4.23)$$

In a similar fashion we obtain that

$$\mathbf{E}_m\left(\left[\int_{(s-\sigma_m)\wedge 0}^{s} \int_{R^1} dN(\tau, x, v)\, \alpha(k(\tau, t)+1)^2 B\right]^2\right) \leqslant C_6 \sigma_m^2 \qquad (4.24)$$

Next we prove (i) for $(y_1(t); t \geqslant 0)$ and we shall do so for $4 > \beta > 2$, $\delta_m = m^\theta$, $1/2 < \theta < \beta/4$, and $\gamma = \beta/4\theta - 1$.

Denote by $\varphi(\tau, x, v; t, s)$ the function

$$\varphi(\tau, x, v; t, s) = \begin{cases} 0 & \text{if } \quad \tau > t \\ \alpha(k^+(\tau, t)+1)\, v_{n(x)} & \text{if } \quad t \geqslant \tau \geqslant s \\ \alpha(k^+(\tau, t) - k^-(\tau, s))\, v_{n(x)} & \text{if } \quad \tau < s \end{cases} \qquad (4.25)$$

We have that

$$\begin{aligned} &\mathbf{E}_m\left(\left|\int_s^t \int_{R^1} dN(\tau, x, v)\, \alpha(k(\tau, t)+1)\, v_{n(x)}\right.\right. \\ &\qquad \left.\left. + \int_0^s \int_{R^1} dN(\tau, x, v)\, \alpha(k(\tau, t) - k(\tau, s))\, v_y\right|^\beta\right) \\ &\leqslant 2^\beta \mathbf{E}_m\left(\left|\int_0^{t \wedge \tau_{B,\delta}} du \int \lambda(u, dx, dv)\, \varphi(u, x, v; t, s)\, \chi(|v_{n(x)}| > 2B)\right|^\beta\right) \\ &\quad + 2^\beta \mathbf{E}_m\left(\left|\int_0^t \int_{R^1} \{dN(\tau, x, v) - d\tau\, \lambda(\tau, dx, dv)\, \chi(\tau < \tau_{B,\delta})\}\right.\right. \\ &\quad \left.\left. \times \varphi(\tau, x, v; t, s)\right|^\beta\right) \end{aligned} \qquad (4.26)$$

using the decomposition of $dN(\tau, x, v)$ into its systematic and martingale parts [cf. (3.16)]. We first handle the systematic part. Using (3.7) and (3.8), we shall show that

$$\begin{aligned} &\mathbf{E}_m\left(\left|\int_0^{t \wedge \tau_{B,\delta}} du \int \lambda(u, dx, dv)\, \varphi(u, x, v; t, s)\, \chi(|v_{n(x)}| > 2B)\right|^\beta\right) \\ &\qquad \leqslant C_7(|t-s| + \sigma_m)^\beta \end{aligned} \qquad (4.27)$$

We establish (4.27) for $t, s \geqslant 2\sigma_m$ and $t, s \leqslant 2\sigma_m$ separately. Then (4.27) for $s \leqslant 2\sigma_m < t$ will follow by observing that

$$\varphi(u, x, v; t, s) = \varphi(u, x, v; t, 2\sigma_m) + \varphi(u, x, v; 2\sigma_m, s)$$

We start with $t, s \geqslant 2\sigma_m$; by (3.6) and (3.8) we get

$$\left| \int_0^{t \wedge \tau_{B,\delta}} du \int \lambda(u, dx, dv)\, \varphi(u, x, v; t, s)\, \chi(|v_{n(x)}| > 2B) \right|$$
$$\leqslant \left| \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du \iint dv\, h_m(v)\, \rho_m \alpha L\, |(v - V_m(u))_x|\, v_x \right|$$
$$+ \left| \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du \iint_{v_y \leqslant -2B} dv\, h_m(v)\, \rho_m \alpha L\, |(v - V_m(u))_y|\, v_y \right.$$
$$+ \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du \iint_{v_y \geqslant 2B} dv\, h_m(v)\, \rho_m (v - V_m(u))_y$$
$$\left. \times \int_{x \in \partial_1} dx\, \chi(x \leqslant 2d(u) \operatorname{tg} v^+ \wedge L)\, \varphi(u, x, v; t, s) \right| \tag{4.28}$$

By symmetry of $h(\cdot)$ [cf. (2.5)] we easily get

$$\left| \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du \iint dv\, h_m(v)\, \rho_m \alpha L\, |(v - V_m(u))_x|\, v_x \right| \leqslant C_8\, |t - s| \tag{4.29}$$

which takes care of the first term on the rhs of (4.28).

For the second term we show that

$$\int_0^{t \wedge \tau_{B,\delta}} du \int_{x \in \partial_1} dx\, \chi(x \leqslant 2d(u) \operatorname{tg} v^+ \wedge L)\, \varphi(u, x, v; t, s)$$
$$= \alpha L v_y (t \wedge \tau_{B,\delta} - s \wedge \tau_{B,\delta}) + O\{m^{3/2} v_y + m^2 v_y^2)[\sigma_m + (t - s)]\} \tag{4.30}$$

Then (4.27) for $t, s > 2\sigma_m$ will follow easily from (4.21) and (4.30).

For simplicity let us assume that $t \leqslant \tau_{B,\delta}$. We show the essential steps. Set

$$a(u, v) = \frac{L v_y}{2d(u)(|v_x| - B)} \left(1 + \frac{C_2}{v_y}\right); \qquad b(u, v) = \frac{(t - s)\, v_y}{2d(u)} \left(1 + \frac{C_2}{v_y}\right)$$

and consider the change of variables

$$x \to \begin{cases} \tilde{x} = \dfrac{v_y}{2d(u)} \dfrac{L - x}{v_x - B} \left(1 + \dfrac{C_2}{v_y}\right) \\[2ex] \hat{x} = \dfrac{v_y}{2d(u)} \dfrac{L - x}{v_x + B} \left(1 - \dfrac{C_1}{v_y}\right) \end{cases}$$

$$u \to \begin{cases} \tilde{u} = \dfrac{v_y}{2d(u)}(t-u)\left(1+\dfrac{C_2}{v_y}\right) \\ \hat{u} = \dfrac{v_y}{2d(u)}(s-u)\left(1-\dfrac{C_1}{v_y}\right) \end{cases}$$

Furthermore, observe that

$$\phi(u, x, v; t, s) = \phi(u, x, v; \infty, s) \equiv \phi_\infty \qquad \text{for} \quad t-u \geqslant \frac{L-x}{v_x - B}$$

$$\phi(u, x, v; t, s) = 0 \qquad \text{for} \quad s-u \geqslant \frac{L-x}{v_x - B}$$

and that

$$|d(u)/d(u') - 1| \leqslant B\delta^{-1}\sigma_m$$

We express equalities to the order given in (4.30) by $\doteq$. The right-hand side of (4.30) is split into

$$\begin{aligned} &\int_0^t du \int_{x\in\partial_1} dx\, \chi\phi \\ &\quad = \int_s^t du \int_{x\in\partial_1} dx\, \chi\phi_\infty + \int_s^t du \int_{x\in\partial_1} dx\, (\phi - \phi_\infty)\chi\, \chi\left(t-u<\frac{L-x}{v_x - B}\right) \\ &\qquad + \int_0^s du \int_{x\in\partial_1} dx\, \chi\phi \end{aligned}$$

which we find, observing (4.25) and Lemma 4.1,

$$\begin{aligned} &\doteq \int_s^t du\, \frac{\alpha L v_y}{a(u,v)} \int_{(a(u,v)-1)\vee 0}^{a(u,v)} dx\, [x+1] \\ &\quad \times \frac{\alpha L^2 v_y}{a(t,v)^2\,(|v_x| - B)} \int_{(a(t,v)-1)\vee 0}^{a(t,v)} dx \int_0^{x\wedge b(t,v)} du\, ([u]-[x]) \\ &\quad \times \frac{\alpha L^2 v_y}{a(s,v)^2\,(|v_x| - B)} \int_{(a(s,v)-1)\vee 0}^{a(s,v)} dx \int_0^x du\, ([x \wedge (\hat{b}(u,v)+u)] - [u]) \end{aligned} \tag{4.31}$$

where $\hat{b}(\cdot, v)$ is the function b under the transformation $u \to \hat{u}$.

The first term on the right is easily computed:

$$\int_s^t du\, \alpha L v_y a(u,v)^{-1} \int_{(a(u,v)-1)\vee 0}^{a(u,v)} dx\, [x+1] = \alpha L v_y (t-s) \tag{4.32}$$

For the next terms observe that for $t-s \geqslant \sigma_m$, $b(t) \geqslant a(t)$ and $\hat{b} \geqslant a(s, v)$, so that both terms transform to

$$\alpha \frac{v_x - B}{v_y} \left\{ \int_{(a(t,v)-1) \vee 0}^{a(t,v)} dx\, d(t)^2 \int_0^x du\, ([u]-[x]) \right.$$
$$\left. - \int_{(a(s,v)-1) \vee 0}^{a(s,v)} dx\, d(s)^2 \int_0^x du\, ([u]-[x]) \right\} \tag{4.33}$$

Now, $a(t, v) \sim 1/d(t)$ and $|(d/dt)[1/d(t)]| \leqslant \delta^{-2}B$, so the mean value theorem gives $[(v_x - B)^{-1} \sim m^{1/2}]$

$$(4.33) \leqslant C\alpha m^{1/2} v_y(t-s)$$

as desired.

For $t-s < \sigma_m$ the third term on the right of (4.31) may be brought into the form

$$\frac{\alpha L^2 v_y}{a(s, v)^2\, (v_x - B)} \left\{ \int_{(a(s,v)-1) \vee 0}^{a(s,v)} dx \int_x^{b(s,v)+x} du\, [x \wedge u] - \int_0^{b(s,v)} du\, [u] \right\}$$

which to the order of interest cancels the second term of (4.31).

Now we shall deal with t, $s < 2\sigma_m$. By (3.6) and (3.8) we have

$$\left| \int_0^{t \wedge \tau_{B,\delta}} du \int_{R^1} \lambda(u, dx, dv)\, \varphi(u, x, v; t, s) \right.$$
$$\leqslant \left| \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du \iint dv\, h_m(v)\, \rho_m \alpha L\, |(v - V_m(u))_x|\, v_x \right|$$
$$+ \left| \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du \iint_{v_y \leqslant -2B} dv\, h_m(v)\, \rho_m \alpha L\, |(v - V_m(u))_y|\, v_y \right.$$
$$+ \int_0^{t \wedge \tau_{B,\delta}} du \iint_{v_y \geqslant 2B} dv\, h_m(v)\, \rho_m\, |(v - V_m(u))_y|$$
$$\times \left\{ \chi(u < \tau_y(u, v)) \int_{x \in \partial_1} dx\, \varphi(u, x, v; t, s) + \chi(u \geqslant \tau_y(u, v)) \right.$$
$$\left. \left. \times \int_{x \in \partial_1} dx\, \chi(x \leqslant 2d(u)\, \mathrm{tg}\, v^+ \wedge L)\, \varphi(u, x, v; t, s) \right\} \right| \tag{4.34}$$

As before, we need only compute the second integral of the second term on the rhs of (4.34). For the sake of simplicity, we shall again assume $t \leqslant \tau_{B,\delta}$; then, since

$$\varphi(u, x, v; t, s) = \varphi(u, x, v; t, 0) - \varphi(u, x, v; s, 0)$$

the same changes of variables used to obtain (4.31) leads to

$$\begin{aligned}
&\int_0^t du \left\{ \chi(u \leqslant \tau_y(u, v)) \int_{x \in \partial_1} dx \, \varphi(u, x, v; t, s) \right. \\
&\qquad \left. + \chi(u < \tau_y(u, v)) \int_{x \in \partial_1} dx \, \chi(x \leqslant 2d(u) \operatorname{tg} v^+ \wedge L) \, \varphi(u, x, v; t, s) \right\} \\
&\quad = \alpha v_y \tau_y(t, v)^2 \, v_x \left\{ \int_{(t/\tau_y(t,v) - 1) \vee 0}^{t/\tau_y(t,v)} du \int_0^{a(t,v)} dx \, ([x \wedge u] + 1) \right. \\
&\qquad \left. + \int_0^{(t/\tau_y(t,v) - 1) \vee 0} du \int_{(a(t,v) - 1) \vee 0}^{a(t,v)} dx \, ([x \wedge u] + 1) \right\} \\
&\qquad + \alpha v_y \tau_y(s, v)^2 \, v_x \left\{ \int_{(s/\tau_y(s,v) - 1) \vee 0}^{s/\tau_y(s,v)} du \int_0^{a(s,v)} dx \, ([x \wedge u] + 1) \right. \\
&\qquad \left. + \int_0^{(s/\tau_y(s,v) - 1) \vee 0} du \int_{(a(s,v) - 1) \vee 0}^{a(s,v)} dx \, ([x \wedge u] + 1) \right\} \\
&\qquad + O(m^{3/2} v_y (t+s)) + o(\sigma_m^3 \alpha v_y^2) \\
&\quad = \alpha v_y L(t-s) + O(m^{3/2} v_y (t+s) + m^{5/2} v_y^2)
\end{aligned} \tag{4.35}$$

To get the last equality, we have used

$$\begin{aligned}
&\int_{(b-1) \vee 0}^{b} du \int_0^a dx \, [x \wedge u] + \int_0^{(b-1) \vee 0} du \int_{(a-1) \vee 0}^{a} dx \, [x \wedge u] \\
&\quad = \int_0^b du \int_0^a dx \, [x \wedge u] - \int_0^{(b-1) \vee 0} du \int_0^{(a-1) \vee 0} dx \, [x \wedge u] \\
&\quad = (ab + 1 - a - b) \, \chi(b > 1, a > 1)
\end{aligned}$$

Thus, (4.27) for $t, s \leqslant 2\sigma_m$ follows easily from (4.34) and (4.35).

Let now consider the martingale part. Once we show that

$$\begin{aligned}
&\mathbf{E}_m \left(\left| \int_0^t \int_{R^1} [dN(\tau, x, v) - d\tau \, \lambda(\tau, dx, dv) \, \chi(\tau \leqslant \tau_{B,\delta})] \, \varphi(\tau, x, v; t, s) \right|^\beta \right) \\
&\quad \leqslant C_{10}((t-s) + \sigma_m)^{\beta/2}
\end{aligned} \tag{4.36}$$

(4.27) and (4.36) will imply (i) with $\gamma = \beta/4\theta - 1$. We observe that

$$\begin{aligned}
M(u) \equiv M_{t,s}(u) = \int_0^u \int & [dN(\tau, x, v) - d\tau \, \lambda(\tau, dx, dv) \, \chi(\tau \leqslant \tau_{B,\delta})] \\
& \times \varphi(\tau, x, v; t, s) \, \chi(|v_{n(x)}| \geqslant 2B)
\end{aligned}$$

is a uniformly integrable $(\mathbf{P}_m, G_t)$-martingale with $M(0) = 0$; then, by the Burkholder–Davis–Gundy inequality we obtain that

$$\mathbf{E}_m(|M(t)|^\beta) \leqslant (4\beta)^\beta \mathbf{E}_m\left(\left[\int_0^t \int_{R^1} dN(u, x, v)\, \phi^2 \chi(u < \tau_{B,\delta})\right]^{\beta/2}\right)$$

$$\leqslant (4\sqrt{2}\,\beta)^\beta \left\{\mathbf{E}_m\left(\left[\int_s^t \int_{R^1} dN(u, x, v)\, \phi^2 \chi(u < \tau_{B,\delta})\right]^{\beta/2}\right)\right.$$

$$\left. + \mathbf{E}_m\left(\left[\int_0^s \int_{R^1} dN(u, x, v)\, \phi^2 \chi(u < \tau_{B,\delta})\right]^{\beta/2}\right)\right\} \tag{4.37}$$

To estimate the right-hand side of (4.37), we may now use Hölder's inequality twice, choosing $p = p'\beta/2$, and obtain

$$\mathbf{E}_m(|M(t)|^\beta) \leqslant (4\sqrt{2}\,\beta)^\beta \left\{\mathbf{E}_m(N_1(s, t)^{\beta q'/2q})^{1/q'}\right.$$

$$\times\ \mathbf{E}_m\left(\left[\int_s^t du \int_{R^1} \lambda(u, dx, dv)\, \phi^{2p} \chi(u < \tau_{B,\delta})\right]^{1/p'}\right)$$

$$+\ \mathbf{E}_m(N_1(s, s - \sigma_m)^{\beta q'/2q})^{1/q'}$$

$$\left.\times\ \mathbf{E}_m\left(\left[\int_0^s du \int_{R^1} \lambda(u, dx, dv)\, \phi^{2p} \chi(u < \tau_{B,\delta})\right]^{1/p}\right)\right\} \tag{4.38}$$

Then, using (4.25) and (3.14) and observing that $t - s > m$, we obtain, for $p > 2$,

$$\mathbf{E}_m(|M(t)|^\beta) \leqslant C_{10}((t-s) + \sigma_m)^{\beta/2} \tag{4.39}$$

and thus (4.36) follows.

Proof of (ii). Let $\{\Delta_n\}_{n=1,M}$, $\Delta_n = [s_{n-1}, s_n)$, be a decomposition of $[0, T]$ in nonoverlapping intervals of length $\delta_m/2 \leqslant |\Delta_n| \leqslant \delta_m$ for all n. By (4.16)–(4.18) we obtain

$$\bar{\mathbf{P}}_m(\{\sup_{\substack{t,s \in [0,T] \\ |t-s| \leqslant \delta_m}} |[y_1(t) - y_1(s)] + [y_2(t) - y_2(s)]| > \varepsilon\})$$

$$\leqslant \mathbf{P}_m(\{\exists n, j\colon t_j \in \Delta_n \text{ and } k(t_j, s_n) > 1\})$$

$$+ \sum_{n=1}^{M} \left[\mathbf{P}_m\left(\left\{\int_{s_{n-1}}^{s_n} \int dN(\tau, x, v) \alpha\, |v_{n(x)}| > \frac{\varepsilon}{3}\right\}\right)\right.$$

$$+ \mathbf{P}_m\left(\left\{\int_{(s_{n-1} - \sigma_m) \vee 0}^{s_n} \int_{R^1} dN(\tau, x, v)\right.\right.$$

$$\left.\left.\times k^+(\tau, s_n) \alpha\, |v_{n(x)}|\, \chi(k^+(\tau, s_n) - k^-(\tau, s_{n-1}) \geqslant 1) > \frac{\varepsilon}{3}\right\}\right)$$

$$+ \mathbf{P}_m\left(\left\{\int_{(s_{n-1} - \sigma_n) \vee 0}^{s_n} \int_{R^1} dN(\tau, x, v)\right.\right.$$

$$\left.\left.\left.\times [\alpha^2 k^+(\tau, s_n) + 1)^2 B] > \frac{\varepsilon}{3}\right\}\right)\right] \tag{4.40}$$

Since $k^+(t_j, s_n) \leqslant \delta_m(1 + C_2/v_y)v_y/2\delta$ for all $t_j \in \Delta_n$, it is obvious that $\mathbf{P}_m(\{\exists n, j\colon t_j \in \Delta_n \text{ and } k(t_j, s_n) > 1\})$ goes to zero as $m \to 0$.

The first and second sums on the right of (4.40) are easily estimated by Chebyschev's inequality using the p correlation function of the point process N for $p > 2$. For the first sum we obtain, for example, the bound $(\theta > 1)$

$$C_{12} m^{-\theta} m^p \left\{ m^\theta m^{-1/2} m^{-1} \int h(v)\, v^2\, dv \right\}^p = C_{12} \Phi_2^p m^{p\theta - \theta - p/2} \tag{4.41}$$

and therefore choosing $p > \theta/(\theta - 1/2)$, one can easily show that the former sum converges to zero as $m \to 0$.

Similarly, one can prove that also the second sum on the rhs of (4.40) converges to zero as $m \to 0$.

For the last sum, observe that $k^+(\tau, s_n) - k^-(\tau, s_n) = 0$ if $|\tau - s_{n-1}| < (L - x_j)(|v_x| - B)^{-1}$ and therefore

$$k^+(\tau, s_n) - k^-(\tau, s_n) \leqslant \frac{L}{2d(\tau)} |v_y|\, \delta_m + (C_1 + C_2) \frac{L}{d(\tau)} (\delta_m + \sigma_m)$$

Then for m small enough

$$\chi(k^+(\tau, s_n) - k^-(\tau, s_{n-1}) \geqslant 1) \leqslant \chi(|v_y| \geqslant 1/2\delta_m)$$

and by Chebyschev's inequality

$$\begin{aligned}
&\sum_{n=1}^{M} \mathbf{P}_m \left(\left\{ \int_{(s_{n-1} - \sigma_m) \vee 0}^{s_n} \int_{R^1} dN(\tau, x, v) \right.\right. \\
&\qquad \left.\left. \times\, \alpha k^+(\tau, s_n) - k^-(\tau, s_{n-1}) \geqslant 1) \geqslant \frac{\varepsilon}{3} \right\} \right) \\
&\qquad \leqslant \frac{\varepsilon}{3} \sum_{n=1}^{M} \mathbf{E}_m \left(\int_{(s_{n-1} - \sigma_m) \vee 0}^{s_n} du \int_{|v_y| > 1/2\delta_m} \lambda(u, dx, dv)\, \alpha k^+(u, s_n)\, |v_y| \right) \\
&\qquad \leqslant C_{13} \delta_m^{-1} \iint_{2\,|v_y| > m^{-\theta + 1/2}} dv\, h(v)\, |v_y|^2
\end{aligned} \tag{4.42}$$

and this concludes the proof of (ii).

Proof of (iii). Part (iii) asserts that "slow" collisions do not contribute in the limit $m \to 0$. The proof of this assertion is not substantially different (for the two-dimensional model) from the one given in ref. 1 if one notes that for $t_j + \sigma_j =$ first time after t_i at which the jth atom reaches the wall we have that $\sigma_j = d(t_j)/|(v_j^+)_y| \geqslant \delta/3B > \sigma_m$, for m small enough.

As stated before (i)–(iii) imply C-tightness in $D([0, \infty), R)$ in the sense of ref. 4. From the proof of Proposition 5.7 in ref. 4 we obtain for our situation even the following result.

Proposition 4.1. Let β and γ be as in Theorem 4.1 and for $\sigma \leqslant 1/2$, $\gamma' < \gamma\sigma/\beta$, and $T < \infty$ let

$$\Omega_m = \{ \sup_{|t-s| < m^\sigma, t, s < \infty} |V_m(t) - V_m(s)| \leqslant m^{\gamma'} \}$$

Then

$$\lim_{m \to 0} \bar{\mathbf{P}}_m(\Omega_m^c) = 0$$

Corollary 4.1. On Ω_m we have that for all $\varepsilon > 0$, $\tau_{B,\delta} \geqslant u \geqslant \sigma_m$ and m sufficiently small

$$\lambda(u, R^1) \geqslant \iint_{|v_y| \geqslant 2B} dv\, h_m(v)\, \rho_m\, |(v - V_m(u))_y| \times \int_{x \in \partial_1} dx\, \chi(x \leqslant 2d(u) \operatorname{tg} v_m(u)^- \wedge L) \tag{4.43a}$$

and

$$\lambda(u, R^1) \leqslant \iint_{|v_y| \geqslant 2B} dv\, h_m(v)\, \rho_m\, |(v - V_m(u))_y| \times \int_{x \in \partial_1} dx\, \chi(x \leqslant 2d(u) \operatorname{tg} v_m(u)^+ \wedge L) \tag{4.43b}$$

where

$$\operatorname{tg} v_m(u)^{+(-)} = \frac{|(v - V_m(u))_x| \pm m^{\gamma'}}{(v + V_m(u))_y - m^{\gamma'}}$$

Furthermore, we have that for any $j \in R^1(0, T)$

$$[\Psi^-(t_j, \zeta_j) \wedge \Psi^-(t_j, t - t_j)] \leqslant k(t_j, t) \leqslant [\Psi^+(t_j, \zeta_j) \wedge \Psi^+(t_j, t - t_j)] \tag{4.44}$$

where $\zeta_j = (L - x_j)\, |(v - V_j^-)_x|^{-1}$ and $\Psi^+(t_j, a)$ and $\Psi^-(t_j, a)$ are the positive and smallest solutions of

$$\frac{2d(t_j)}{|(v_j^- - V_j^-)_y|} \left\{ 1 + \frac{2V_j^- \pm C^\pm m^{\gamma'}}{|(v_j^- - V_j^-)_y|} \Psi^\pm \right\} \Psi^\pm = a \tag{4.45}$$

$C^\pm$ are positive constants.

Proof. To get (4.42)–(4.44) one can proceed as in (3.7), (3.8), and Lemma 4.1, respectively, observing that on Ω_m, $|V_{j,k}^- - V_j^-| \leqslant m^{\gamma'}$, since for all $j \in R^1(0, T)$ and $k \leqslant k(t_j, T)$, $|t_{j,k} - t_j| \leqslant 2\sigma_m$.

5. LIMIT PROCESS

To establish Theorem 2.1, we need only prove the uniqueness of the limit measure. For this we use the martingale characterization of diffusion processes by Stroock and Varadhan. This amounts to showing the following.

Theorem 5.1. For any $f \in C_0^\infty(\mathbf{R}^2)$, $g_1, g_2,..., g_k \in C_0^\infty(\mathbf{R}^4)$, and $u_1 < u_2 < \cdots < u_k < s < t$

$$\lim_{m \to 0} \mathbf{E}_m \left(\left\{ f(\bar{V}_m(t)) - f(\bar{V}_m(s)) - \int_0^t du\, (\mathscr{L}f)(\bar{X}_m(u), \bar{V}_m(u)) \right\} \prod_{i=1}^k g_i(\bar{X}_m(u_i), \bar{V}_m(u_i)) \right) = 0$$

where $\mathscr{L}$ is the operator

$$(\mathscr{L}f)(X, V) = -b(X, V)(\nabla_v f)(V) + 1/2[\sigma(X)\nabla_v]^2 f(V)$$

and $\nabla_v = (\partial/\partial v_x, \partial/\partial v_y)$.

We define a family of processes $(\tilde{V}_m(t); t \geqslant 0)$ approximating $(\bar{V}_m(t); t \geqslant 0)$. For any $a > 0$ and $j \in R^1(0, \infty)$, set $(\Delta v_j)_n = (v_j^- - V_j^-)_n$, and let $\varphi(t_j, a)$ be the smallest and positive solution of the following equation [cf. Eq. (4.45) for $C^\pm = 0$]:

$$\frac{2d(t_j)}{(\Delta v_j)_y} \left[1 + \frac{2V_j^-}{(\Delta v_j)_y} \varphi \right] \varphi = a \tag{5.1}$$

Then the process $(\tilde{V}_m(t); t \geqslant 0)$ is defined as follows:

$$\tilde{V}_m(t) = V_0 + \int_0^t \int_{R_i} dN(\tau, x, v) \{ \alpha(\tilde{k}(\tau, t) + 1)(\Delta v)_n - \alpha(\tilde{k}(\tau, 1) + 1)\, V_m(\tau^-) \}$$

$$\equiv V_0 + \int_0^t \int_{R_i} dN(\tau, x, v)\, W(\tau, x, v; t) \tag{5.2}$$

where

$$\tilde{k}(t_j, t) = [\varphi(t_j, \zeta_j) \wedge \varphi(t_j, t - t_j)] \qquad \text{if} \quad t_j < t,\ x_j \in \partial_1$$

$$\tilde{k}(t_j, t) = 0 \qquad \text{if} \quad t_j > t \text{ or } x_j \notin \partial_1$$

$$\zeta_j = (L - x_j)(\Delta v_j)_x^{-1}$$

Note that dN above integrates only predictable functions.

We shall use $\tilde{V}_m$, $\tilde{X}_m = \int_0 \tilde{V}_m + X_0$ when computing expectations involving $\bar{V}_m$, $\bar{X}_m$. For this we prove the following.

Theorem 5.2. For any $t < \infty$

$$\lim_{m \to 0} \mathbf{E}_m(|\bar{V}_m(t) - \tilde{V}_m(t)|) = 0 \tag{5.2a}$$

$$\lim_{m \to 0} \mathbf{E}_m(|\bar{X}_m(t) - \tilde{X}_m(t)|) = 0 \tag{5.2b}$$

Proof. We need only show (5.2a). We have

$$\begin{aligned} \bar{V}_m(t) - \tilde{V}_m(t) = &\sum_{j \in R^1(0,t)} \left\{ \sum_{k=0}^{k(t_j,t)} (V_{j,k}^+ - V_{j,k}^-) - W(t_j, x_j, v_j; t) \right\} \\ &+ \sum_{j \in R^2(0,t)} \sum_{k=0}^{k(t_j,t)} (V_{j,k}^+ - V_{j,k}^-) \end{aligned} \tag{5.3}$$

From (iii) of Theorem 4.1 it follows easily that

$$\lim_{m \to 0} \mathbf{E}_m \left(\left| \sum_{j \in R^2(0,t)} \sum_{k=0}^{k(t_j,t)} (V_{j,k}^+ - V_{j,k}^-) \right| \right) = 0 \tag{5.4}$$

Furthermore, similarly as in (4.16) and (4.17), we have that for all $j \in R^1(0, T)$ and $k \leqslant k(t_j, T)$

$$V_{j,k}^+ - V_{j,k}^- = \alpha(\Delta v_j)_y + 2\alpha k(V_j^-)_y + \alpha f^*(v_j^-, V_{j,k}^-, \ldots, V_j^-) \tag{5.5}$$

where

$$\begin{aligned} f^*(v_j^-, V_{j,k}^-, \ldots, V_j^-) = &[1 - (1-\alpha)^k](\Delta v_j)_y + \alpha \sum_{i=0}^{k-1} (1-\alpha)^i (V_{j,k-i-1}^-)_y \\ &- 2 \sum_{i=0, k \geqslant 1}^{k-1} (1-\alpha)^i (V_{j,k-i-1}^- - V_j^-)_y \\ &- 2\alpha^{-1}[1 - \alpha k - (1-\alpha)^k](v_j^-)_y \end{aligned} \tag{5.6}$$

It is easy to check that

$$\begin{aligned} &\left| \sum_{k=0}^{k(t_j,t)} f^*(v_j^-, V_{j,k}^-, \ldots, V_j^-) \right| \\ &\quad \leqslant \alpha k^+(t_j, t)[|(v_j^-)_y| + 2B] \\ &\qquad + 2k^+(t_j, t)^2 \sup_{k \leqslant k(t_j,t)} \{|V_{j,k}^- - V_j^-|\} + \alpha^2 B k^+(t_j, t) \end{aligned} \tag{5.7}$$

Hence, by virtue of Proposition 4.1, (5.5)–(5.7) yield for the first sum in (5.3)

$$\lim_{m\to 0} \mathbf{E}_m\left(\left|\sum_{j\in R^1(0,t)} \sum_{k=0}^{k(t_j,t)} (V_{j,k}^+ - V_{j,k}^-) - W(t_j, x_j, v_j)\right|\right)$$

$$\leqslant \lim_{m\to 0} \mathbf{E}_m\left(\int_0^t \int_{R^1} dN(\tau, x, v)\right.$$

$$\left.\times\, |\alpha(k(\tau,t) - \tilde{k}(\tau,t))(v + V_m(\tau^-))_y + \alpha(k(\tau,t)^2 - \tilde{k}(\tau,t)^2)\, V_m(\tau^-)_y|\right)$$

$$+ \lim_{m\to 0} \mathbf{E}_m\left(\int_0^t \int_{R^1} dN(\tau, x, v)\right.$$

$$\left.\times\, [\alpha^2 k^+(\tau,t)(v_y + 2B) + 2\alpha k^+(\tau,t)^2 m^\gamma + 3B\alpha^3 k^+(\tau,t)^3]\right) \tag{5.8}$$

The second term on the right of (5.8) is easily shown to be zero. For the first limit on the rhs of (5.8), note that the definition of $\Psi^\pm(t_j, \zeta_j)$ of (4.45) implies that

$$0 \leqslant \Psi^+(t_j,\zeta_j) - \Psi^-(t_j,\zeta_j) \leqslant 2(C^+ + C^-)\, m^{\gamma'} \zeta_j^2 (\Delta v_j)_y\, (2\delta)^{-2}$$

Therefore, if $\Delta v_y/\Delta v_x^2 \leqslant m^{-\gamma'/2}$, then $k(t_j, t)$ and $\tilde{k}(t_j, t)$ may differ at most by one for m sufficiently small; this is possible only if there exists an integer n such that

$$|n - \varphi(t_j,\zeta_j) \wedge \varphi(t_j, t - t_j)| \leqslant \frac{L^2(C^+ + C^-)(\Delta v_j)_y}{\delta^2 (\Delta v_j)_x^2}\, m^{\gamma'}$$

$$\leqslant \frac{L^2(C^+ + C^-)}{\delta^2}\, m^{\gamma'/2} \equiv C_m$$

Thus, for all $j \in R^1(0, t)$,

$$\left\{k(t_j,t) \neq \tilde{k}(t_j,t) \text{ and } \frac{(\Delta v_j)_y}{(\Delta v_j)_x^2} \leqslant m^{-\gamma'/2}\right\} \subseteq \bigcup_{n=0}^{N(v_j^-)} \{\zeta_j \wedge (t - t_j) \in I_n\} \tag{5.9}$$

where $N(v_j^-) = [L(v_j^-)_y/\delta\, |(v_j)_x|]$ and I_n is the interval

$$\left[2d(t_j)\frac{1}{|(\Delta v_j)_y|}(n - C_m)\left\{1 + \frac{2V_j^- - Cm^{\gamma'}}{(\Delta v_j)_y}(n - C_m)\right\},\right.$$

$$\left. 2d(t_j)\frac{1}{|(\Delta v_j)_y|}(n - C_m)\left\{1 + \frac{2V_j^- + Cm^{\gamma'}}{(\Delta v_j)_y}(n - C_m)\right\}\right] \tag{5.10}$$

with length

$$|I_n| \leqslant D_1 \frac{d(t_j)\, m^{\gamma'}}{(\Delta v_j)_y^2} \tag{5.11}$$

Therefore, the first limit on the right of (5.8) is bounded by

$$\begin{aligned}
&\mathbf{E}_m\Bigg(\chi(\exists j \in R^1(0, T)\colon (\Delta v_j)_y\, (\Delta v_j)_x^{-2} \geqslant m^{-\gamma'/2}) \\
&\quad \times \int_0^t \int_{R^1} dN(\tau, x, v)\, \alpha(2k^+(\tau, t)\, v_y B + 2k^+(\tau, t)^2\, B)\Bigg) \\
&\quad + \mathbf{E}_m\Bigg(\left(\int_0^t \int_{R^1} dN(\tau, x, v)\, \alpha(|v_y| + 2B[k^+(\tau, t) + 1])\right) \\
&\quad \times \sum_{n=0}^{N(v)} \chi\left(\frac{L-x}{|\Delta v_x|} \wedge (t-\tau) \in I_n\right)\Bigg)
\end{aligned}$$

The first term is easily shown to converge to zero using Schwartz's inequality and observing that, by (2.7),

$$\mathbf{E}_m(\chi\{\cdot\}) \leqslant D_2 \int dv\, h(v)\, v_y \chi(v_y \geqslant v^2/2m^{1/2+\gamma'/2}) \leqslant D_3 m^2 \tag{5.12}$$

By virtue of (5.11) we obtain for the second term the bound

$$\begin{aligned}
&D_4 \rho_m m^{\gamma'} \int_{v_y \geqslant 2B} dv \\
&\qquad \times h_m(v)\, v_y[v_y + 2B(L\delta^{-1}\, \mathrm{tg}\, v + 1)]\, \mathrm{tg}\, v\, \Delta v_x/\Delta v_y^2 \\
&\quad \leqslant D_5 m^{\gamma'}
\end{aligned}$$

where $\mathrm{tg}\, v = v_y/v_x$ and D_2 through D_5 are constants.

We have thus established (5.2a).

Proof of Theorem 5.1. By Theorem 5.2, Theorem 5.1 will follow once we show that

$$\begin{aligned}
&\lim_{m \to 0} \mathbf{E}_m\Bigg(\left\{f(\tilde{V}_m(t)) - f(\tilde{V}_m(s)) - \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du\, (\mathscr{L}f)(\tilde{X}_m(u), \tilde{V}_m(u))\right\} \\
&\quad \times \prod_{i=1}^k g_i(\tilde{X}_m(u_i), \tilde{V}_m(u_i))\Bigg) = 0
\end{aligned} \tag{5.13}$$

Stroock and Varadhan (ref. 6, Corollary 4.2.2 and Exercise 4.6.6) showed that we need only prove (5.13) for the functions $f_1(V) = V_x + V_y$, $f_2(V) = V_x V_y$, and $f_3(V) = V_x^2 + V_y^2$. We shall begin by proving (5.13) for $f = f_1$, i.e.,

$$\lim_{m \to 0} \mathbf{E}_m \Bigg(\Bigg\{ f_1(\tilde{V}_m(t)) - f_1(\tilde{V}_m(s)) - \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du \, (b_x \tilde{V}_m(u)_x + b_y(\tilde{X}_m(u), V_m(u)) \, \tilde{V}_m(u)_y) \Bigg\} \times \prod_{i=1}^{k} g_i(\tilde{X}_m(u_i), \tilde{V}_m(u_i)) \Bigg) = 0 \tag{5.14}$$

The definition of $\tilde{V}_m(t)$ implies that $\tilde{V}_m(s) = \tilde{V}_m(t)$ for $s > \tau_{B,\delta}$ and for $s < \tau_{B,\delta}$. We have

$$\begin{aligned} f_1(\tilde{V}_m(t)) - f_1(\tilde{V}_m(s)) &= \int_s^t \int dN(\tau, x, v) \alpha \, \Delta v_x \, \chi(x \in \tilde{\partial}_2) \\ &\quad + \int_0^t \int dN(\tau, x, v) [W(\tau, x, v; t) - W(\tau, x, v; s)]_y \, \chi(x \in \tilde{\partial}_1, |v_y| > 2B) \end{aligned} \tag{5.15}$$

where $\tilde{\partial}_1$ and $\tilde{\partial}_2$ denote the boundaries of the molecule parallel and orthogonal to the wall, respectively. One easily computes that

$$\begin{aligned} \lim_{m \to 0} \mathbf{E}_m \left(\int_s^t \int dN(\tau, x, v) \alpha \, \Delta v_x \, \chi(x \in \tilde{\partial}_2) \prod g_i \right) &= \lim_{m \to 0} \mathbf{E}_m \left(- \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du \, b V_m(u^-)_x \prod g_i \right) \\ &= \lim_{m \to 0} \mathbf{E}_m \left(- \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du \, b \tilde{V}_m(u^-)_x \prod g_i \right) \end{aligned} \tag{5.16}$$

where we have used in the last equality Theorem 5.2 and

$$\prod g_i = \prod_{i=1}^{k} g_i(\tilde{X}_m(u_i), \tilde{V}_m(u_i))$$

Because of Theorem 5.2 and Corollary 4.1, we need only evaluate the expectation of the second integral on the rhs of (5.15) on Ω_m.

Furthermore, we may assume that $s > \sigma_m$, $t - s > \sigma_m$, and $u_k < s - \sigma_m$.

By Corollary 4.2 we obtain, observing (3.16), that

$$\lim_{m\to 0} \mathbf{E}_m\left(\int_0^t \int dN(\tau, x, v)[W(\tau, x, v; t) - W(\tau, x, v; s)]_y \right.$$

$$\left. \times \chi(x \in \tilde{\partial}_1, |v_y| > 2B) \prod g_i\right)$$

$$= \lim_{m\to 0} \mathbf{E}_m\left(\left\{ - \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du\, \rho_m \alpha L \iint_{v_y < -2B} dv\, h_m(v)[-v_y^2 + 2V_m(u^-)_y v_y] \right.\right.$$

$$+ \int_0^{t \wedge \tau_{B,\delta}} du\, \rho_m \alpha \iint_{v_y > 2B} dv\, h_m(v)[v - V_m(u^-)]_y$$

$$\times \int_{x\in\partial_1} dx\, [W(u, x, v; t) - W(u, x, v; s)]_y$$

$$\left.\left. \times \chi(x \leqslant 2d(u) \operatorname{tg} v_m(u)^+ \wedge L)\right\} \prod g_i\right) \tag{5.17}$$

From now on, for the sake of simplicity, we shall denote by $\approx$ equalities which become true in the limit $m \to 0$. For any $s, t > \sigma_m$, we have that

$$\begin{aligned} &W(u, x, v; t) - W(u, x, v; s) \\ &\quad = W(u, x, v; \infty)\, \chi(u \in [s, t]) \\ &\qquad - [W(u, x, v; \infty) - W(u, x, v; t)]\, \chi(u > t - \sigma_m) \\ &\qquad + [W(u, x, v; t) - W(u, x, v; s)]\, \chi(u \in [s - \sigma_m, s]) \end{aligned} \tag{5.18}$$

Next we prove that

$$\mathbf{E}_m\left(\left\{\int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du\, \rho_m \alpha L \iint_{|v_y| > 2B} dv\, h_m(v)\, |[v - V_m(u^-)]_y| \right.\right.$$

$$\left.\left. \times \int_{x\in\partial_1} dx\, W(u, x, v; \infty)_y \right\} \prod g_i\right)$$

$$= \mathbf{E}_m\left(- \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du\, b_y V_m(u^-)_y \right.$$

$$\times \left\{ 1 + \Phi_{1,y}^{-1} \iint_{v_y > 0} dv\, h(v) \left[\frac{\partial}{\partial v_y} v_y^2 F(\eta(X_m(u), v)) \right.\right.$$

$$\left.\left.\left. + v_y (F'(\eta(X_m(u), v))\, \eta(X_m(u), v)) - F(\eta(X_m(u), v)) \right] \prod g_i \right\}\right) \tag{5.19}$$

and that

$$\mathbf{E}_m\left(\left\{\int_0^{t\wedge\tau_{B,\delta}} du\,\rho_m\alpha L\iint_{v_y>2B} dv\,h_m(v)[v-V_m(u^-)]_y\right.\right.$$
$$\times\int_{x\in\partial_1} dx\,[2W(u,x,v;t)-W(u,x,v;\infty)-W(u,x,v;s)]_y$$
$$\left.\left.\times\chi(x\leqslant 2d(u)\,\mathrm{tg}\,v_m(u)^+\wedge L)\right\}\prod g_i\right)$$
$$\approx\mathbf{E}_m\left(\rho_m\alpha L^2\iint_{v_y>2B} dv\,h_m(v)\,v_y\right.$$
$$\left.\times\{2d(s)\,F(\eta(X_m(s\wedge\tau_{B,\delta}),v))-2d(t)\,F(\eta(X_m(t\wedge\tau_{B,\delta}),v))\}\prod g_i\right) \tag{5.20}$$

then (5.14) will follow from (5.16)–(5.20) by observing that

$$\frac{d}{du}\mathbf{E}_m(2d(u)\,F(\eta(X_m(u),v)))$$
$$=\mathbf{E}_m(2V_m(u^-)[F(\eta(X_m(u),v))-F'(\eta(X_m(u),v))\,\eta(X_m(u),v)]) \tag{5.21}$$

For (5.19), by (4.2) and (4.3) of Corollary 4.2, we obtain that

$$\mathbf{E}_m\left(\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,\rho_m\alpha\iint_{v_y>2B} dv\,h_m(v)[v-V_m(u^-)]_y\right.$$
$$\left.\times\int_{x\in\partial_1} dx\,W(u,x,v;\infty)_y\,\chi(x\leqslant 2d(u)\,\mathrm{tg}\,v_m(u)^+\wedge L)\prod g_i\right)$$
$$\approx\mathbf{E}_m\left(\int_s^t du\,\rho_m\alpha\iint_{v_y>2B} dv\,h_m(v)\int_{x\in\partial_1} dx\,[\tilde{k}(u,\infty)+1]\right.$$
$$\left.\times\{v_y^2-2v_yV_m(u^-)_y-\tilde{k}(u,\infty)\,v_yV_m(u^-)_y\}\,\chi(x\leqslant 2d(u)\,\mathrm{tg}\,v(u)\wedge L)\prod g_i\right) \tag{5.22}$$

where $\mathrm{tg}\,v(u)=|\Delta v_x|\,[v_y+V_m(u^-)_y]^{-1}$.

Set

$$\alpha(u,v)=\frac{L\,\Delta v_y}{2d(u)\,\Delta v_x},\qquad \eta_0(u,v)=\varphi\left(u,\frac{L}{|\Delta v_x|}\right)$$
$$\eta_1(u,v)=\varphi\left(u,L-\frac{2d(u)\,\mathrm{tg}\,v(u)}{|\Delta v_x|}\right)$$

then the change of variable $x \to \eta = \varphi(u, (L-x)\,|\Delta v_x|^{-1})$ leads to

$$\int_{x \in \partial_1} dx\, [\tilde{k}(u, \infty) + 1]\, \chi(x \leqslant 2d(u) \operatorname{tg} v(u) \wedge L)$$

$$= 2d(u) \frac{|\Delta v_x|}{\Delta v_y} \Bigg\{ \chi(2d(u) \operatorname{tg} v(u) > L) \times \int_0^{\eta_0(u,v)} d\eta \left(1 + \frac{4V_m(u^-)_y}{\Delta v_y}\eta\right)[\eta + 1] + \chi(2d(u) \operatorname{tg} v(u) \leqslant L) \int_{\eta_1(u,v)}^{\eta_0(u,v)} d\eta \left(1 + \frac{4V_m(u^-)_y}{\Delta v_y}\eta\right)[\eta+1] \Bigg\} \tag{5.23}$$

By Taylor expansion we obtain that

$$|\eta_0(u, v) - \alpha(u, v)| \leqslant D_3 v_y v_x^{-2} \tag{5.24}$$

and

$$|[\eta_0(u, v) - 1] - \eta_1(u, v) + 4V_m(u^-)_y\, [v_y + V_m(u^-)_y]^{-1}\, \alpha(u, v)| \leqslant D_4(1 + v_y^3\, |v_x|^{-3})/v_y^2 \tag{5.25}$$

D_3, D_4 are positive constants.

Since $\eta_0(u, v) = \alpha(u, v) - 2V_m(u^-)_y\, \Delta v_y^{-1}\, \eta_0(u, v)^2$, by (5.24), we obtain that for m sufficiently small the first integral on the rhs of (5.23) is equal to L. For the second integral of (5.25) we have that

$$\int_{\eta_1(u,v)}^{\eta_0(u,v)} d\eta \left(1 + \frac{4V_m(u^-)_y}{\Delta v_y}\eta\right)[\eta+1]$$

$$= \int_{\eta_0(u,v)-1}^{\eta_0(u,v)} d\eta \left(1 + \frac{4V_m(u^-)_y}{\Delta v_y}\eta\right)[\eta+1] - \frac{4V_m(u^-)_y}{v_y - V_m(u^-)_y}\, \alpha(u, v)[\eta_0(u, v)] + \int_{\eta_1(u,v)}^{\eta_0(u,v)-1} d\eta\, ([\eta+1] - [\eta_0(u, v)]) + o(\eta_0(u, v)^3\, (\Delta v_y)^{-2}) \tag{5.26}$$

By (5.25), $|[\eta + 1] - \eta_0(u, v)| \leqslant 1$ for $\eta \in [\eta_1(u, v), \eta_0(u, v) - 1]$ and B large enough; therefore, again by (5.25),

$$\lim_{m \to 0} m^{1/2} \iint_{v_y > 2B} dv\, h_m(v)[v_y^2 - 2V_m(u^-)_y]\, 2d(u) \frac{|\Delta v_x|}{\Delta v_y} \times \int_{\eta_1(u,v)}^{\eta_0(u,v)-1} d\eta\, |[\eta+1] - \eta_0(u, v)|$$

$$\leqslant \lim_{m \to 0} m^{1/2} \iint_{v_y > 2B} dv\, h_m(v)\, v_y 4B\chi([\eta_1(u, v)] \neq [\eta_0(u, v) - 1]) = 0 \tag{5.27}$$

Furthermore, since $\int_{\eta_0-1}^{\eta_0} d\eta\,[\eta]\,\eta=(\eta_0-1)^2/2+\eta_0 F(\eta_0)$, where $F(\cdot)$ is the function defined in Theorem 2.1, then we have that

$$\int_{\eta_0(u,v)-1}^{\eta_0(u,v)} d\eta \left(1+\frac{4V_m(u^-)_y}{\Delta v_y}\eta\right)[\eta+1]-\frac{4V_m(u^-)_y}{v_y-V_m(u^-)_y}\alpha(u,v)[\eta_0(u,v)]$$
$$=\alpha(u,v)+4V_m(u^-)_y\,\Delta v_y^{-1}\,\eta_0(u,v)\,F(\eta_0(u,v))-2\alpha(u,v)\,2V_m(u^-)_y$$
$$\times[v_y+V_m(u^-)_y]^{-1}\,[\eta_0(u,v)\,F'(\eta_0(u,v))+F(\eta_0(u,v))] \tag{5.28}$$

where $F'(\cdot)$ denotes the first derivative of $F(\cdot)$.

Finally, by (5.23) and (5.26)–(5.28) we obtain

$$\mathbf{E}_m\Bigg(\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,\rho_m\alpha\iint_{v_y>2B} dv\,h_m(v)\int_{x\in\partial_1} dx\,[\tilde{k}(u,\ )+1][v_y^2-2v_y V_m(u^-)_y]$$
$$\times\chi(x\leqslant 2d(u)\operatorname{tg} v_m(u)^+\wedge L)\prod g_i\Bigg)$$
$$\approx\mathbf{E}_m\Bigg(\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,\rho_m\alpha L\iint_{v_y>2B} dv\,h_m(v)$$
$$\times\{v_y^2-2V_m(u^-)_y\,v_y-4V_m(u^-)_y\,v_y\eta_0(u,v)\,F'(\eta_0(u,v))\}\prod g_i\Bigg) \tag{5.29}$$

Similarly as before, i.e., using the same change of variable and observing that $\int_{\eta_0-1}^{\eta_0} d\eta\,[\eta][\eta+1]=2\eta_0 F(\eta_0)$, we can show that

$$\mathbf{E}_m\Bigg(-\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,\rho_m\alpha\iint_{v_y>2B} dv\,h_m(v)\int_{x\in\partial_1} dx\,[\tilde{k}(u,\infty)+1]$$
$$\times\tilde{k}(u,\infty)\,V_m(u^-)_y\,v_y\chi(x\leqslant 2d(u)\operatorname{tg} v_m(u)^+\wedge L)\prod g_i\Bigg)$$
$$\approx\mathbf{E}_m\Bigg(-\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,\rho_m\alpha L\iint_{v_y>2B} dv\,h_m(v)\,2V_m(u^-)_y\,v_y F(\eta_0(u,v))\prod g_i\Bigg) \tag{5.30}$$

Hence (5.17), (5.29), and (5.30) imply

$$\lim_{m\to 0}\mathbf{E}_m\left(\int_s^t\int dN(\tau,x,v)\,W(\tau,x,v;\infty)\,\chi(x\in\tilde{\partial}_1,|v_y|>2B)\prod g_i\right)$$
$$=\lim_{m\to 0}\mathbf{E}_m\Bigg(-\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,b_y V_m(u^-)_y\Bigg\{1+\phi_{1,y}^{-1}\iint_{v_y>0} dv\,h(v)\,v_y$$
$$\times[2F'(\eta_0(u,m^{-1/2}v))\,\eta_0(u,m^{-1/2}v)+F(\eta_0(u,m^{-1/2}v))]\Bigg\}\prod g_i\Bigg) \tag{5.31}$$

Then (5.19) will be an easy consequence of (5.13) upon observing that

$$\frac{\partial}{\partial v_y}[v_y^2 F(\eta(X, v))] = v_y[F'(\eta(X, v))\,\eta(X, v) + 2F(\eta(X, v))]$$

where $\eta(X_m(u), v) = Lv_y/2d(u)\,|v_x|$ (cf. Theorem 2.1).

To prove (5.20), we first note that

$$\begin{aligned}
&\mathbf{E}_m\left(\int_{t-\sigma_m}^{t}\int dN(\tau, x, v)[W(\tau, x, v; \infty) - W(\tau, x, v; t)]_y\right.\\
&\qquad\left.\times\chi(x \in \partial_1, |v_y| > 2B)\prod g_i\right)\\
&\qquad + \mathbf{E}_m\left(\int_{s-\sigma_m}^{s}\int dN(\tau, x, v)[W(\tau, x, v; t) - W(\tau, x, v; s)]_y\right.\\
&\qquad\left.\times\chi(x \in \partial_1, |v_y| > 2B)\prod g_i\right)\\
&\approx \mathbf{E}_m\left(\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,\rho_m\alpha\iint_{v_y > 2B} dv\,h_m(v)\int_{x\in\partial_1} dx\,[\tilde{k}(u, \infty) - \tilde{k}(u, t)]\right.\\
&\qquad\left.\times v_y^2\chi(x \leqslant 2d(u)\,\mathrm{tg}\,v(u) \wedge L)\prod g_i\right)\\
&\qquad + \mathbf{E}_m\left(\int_0^{s\wedge\tau_{B,\delta}} du\,\rho_m\alpha\iint_{v_y > 2B} dv\,h_m(v)\int_{x\in\partial_1} dx\,[\tilde{k}(u, t) - \tilde{k}(u, s)]\right.\\
&\qquad\left.\times v_y^2\chi(x \leqslant 2d(u)\,\mathrm{tg}\,v(u) \wedge L)\prod g_i\right)
\end{aligned}\tag{5.32}$$

Then, using the change of variables

$$x \to \eta = \varphi(u, (L - x)\,|\Delta v_x|^{-1}), \qquad u \to \sigma = \varphi(u, t - u)$$

and proceeding in the same way as in (4.31) and (4.33), we can easily get (5.20).

Now let us consider the case $f = f_2$; we have to prove that

$$\begin{aligned}
&\mathbf{E}_m\left(\left\{f_2(\tilde{V}_m(t)) - f_2(\tilde{V}_m(s))\right.\right.\\
&\left.\left.+\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,b_x\tilde{V}_m(u^-)_x\,\tilde{V}_m(u^-)_y + b_y(\tilde{X}_m(u), \tilde{V}_m(u^-))\,\tilde{V}_m(u^-)_x\right\}\prod g_i\right) = 0
\end{aligned}$$

For simplicity we put $\tilde{V}_0 = 0$. We introduce for all $0 < u' < u$

$$G(u', u) = \int_0^{u'} \int dN(\tau, x, v)\ W(\tau, x, v; \infty)\ \chi(|v_{n(x)}| > 2B)$$

and we obtain that

$$\begin{aligned}
&f_2(\tilde{V}_m(t)) - f_2(\tilde{V}_m(s)) \\
&\quad = \iint_0^t dN(\tau, x, v)\ W(\tau, x, v; \infty)_y\ \tilde{V}_m(\tau^-)_x\ \chi(x \in \tilde{\partial}_1, |v_y| > 2B) \\
&\qquad + \iint_s^t dN(\tau, x, v) \alpha\ \Delta v_x\ \tilde{V}_m(\tau^-)_y\ \chi(x \in \tilde{\partial}_2) \\
&\qquad + \iint_{t-\sigma_m}^t dN(\tau, x, v)[W(\tau, x, v; t) - W(\tau, x, v; \infty)]_y\ \tilde{V}_m(\tau^-)_x\ \chi(|v_y| > 2B) \\
&\qquad + \iint_{s-\sigma_m}^s dN(\tau, x, v)[W(\tau, x, v; \infty) - W(\tau, x, v; s)]_y\ \tilde{V}_m(\tau^-)_x\ \chi(|v_y| > 2B) \\
&\qquad + \iint_s^t dN(\tau, x, v) \alpha\ \Delta v_x\ [G(\tau, t) - \tilde{V}_m(\tau^-)]_y\ \chi(x \in \tilde{\partial}_2) \\
&\qquad + \iint_{s-\sigma_m}^s dN(\tau, x, v) \alpha\ \Delta v_x\ [G(\tau, t) - G(\tau, s)]_y\ \chi(x \in \tilde{\partial}_2)
\end{aligned} \tag{5.33}$$

Proceeding as (5.16) and (5.31), we get that

$$\begin{aligned}
&\mathbf{E}_m \Bigg(\Bigg\{ \iint_s^t dN(\tau, x, v) \alpha\ \Delta v_x\ \tilde{V}_m(\tau^-)_y\ \chi(x \in \tilde{\partial}_2) \\
&\qquad + \iint_s^t dN(\tau, x, v) \alpha\ \Delta v_y\ \tilde{V}_m(\tau^-)_x\ \chi(x \in \tilde{\partial}_2) \Bigg\} \prod g_i \Bigg) \\
&\qquad \approx \mathbf{E}_m \Bigg(- \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du\ [b_x V_m(u^-)_y\ \tilde{V}_m(u^-)_y \\
&\quad + b_y(X_m(u))\ V_m(u^-)_y\ \tilde{V}_m(u^-)_x] \prod g_i \Bigg) \\
&\qquad + \mathbf{E}_m \Bigg(\int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du\ 2\alpha\rho_m L V_m(u^-)_y\ \tilde{V}_m(u^-)_y \\
&\qquad \times \langle v_y \eta_0(u, v)\ F'(\eta_0(u, v)) - F(\eta_0(u, v)) \rangle \prod g_i \Bigg)
\end{aligned} \tag{5.34}$$

where $\eta_0(u, v)$ is the function defined in (5.26) and $\langle (\cdot) \rangle$ denotes the mean with respect to the measure $h_m(v)\, dv$.

In a similar fashion as in (5.30), one can easily show that

$$\mathbf{E}_m\Bigg(\Bigg\{\iint_{t-\sigma_m}^{t} dN(\tau,x,v)[W(\tau,x,v;t)-W(\tau,x,v;\infty)]_y\,\tilde{V}_m(\tau^-)_x\,\chi(|v_y|>2B)$$

$$+\iint_{s-\sigma_m}^{s} dN(\tau,x,v)[W(\tau,x,v;\infty)-W(\tau,x,v;s)]_y$$

$$\times\,\tilde{V}_m(\tau^-)_x\,\chi(|v_y|>2B)\Bigg\}\prod g_i\Bigg)$$

$$\approx\mathbf{E}_m\Big(\{-2\rho_m\alpha L\tilde{V}_m(t^-)_x\,d(t)\langle v_yF(\eta(X_m(t\wedge\tau_{B,\delta}),v))\rangle$$

$$-\,\tilde{V}_m(s^-)_x\,d(s)\langle v_yF(\eta(X_m(s\wedge\tau_{B,\delta}),v))\rangle\}\prod g_i\Big) \tag{5.35}$$

and, by Corollary 4.1,

$$\lim_{m\to0}\mathbf{E}_m\left(\int_s^t\int dN(\tau,x,v)\alpha\,\Delta v_x\,[G(\tau,t)-\tilde{V}_m(\tau^-)]_y\,\chi(x\in\tilde{\partial}_2)\prod g_i\right)$$

$$=\lim_{m\to0}\mathbf{E}_m\Bigg(\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,2\rho LM^{-1}\Phi_{1,x}$$

$$\times\iint_{u-\sigma_m}^{u} dN(\tau,x,v)[W(\tau,x,v;t)-W(\tau,x,v;u)]_y\,V_m(\tau^-)_x\prod g_i\Bigg)$$

$$=\lim_{m\to0}\mathbf{E}_m\Bigg(\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,2\rho LM^{-1}\Phi_{1,x}\,V_m(u^-)_x\,2\rho_m\alpha Ld(u)$$

$$\times\langle v_yF(\eta(X_m(u),v))\rangle\prod g_i\Bigg) \tag{5.36}$$

Note that $d\tilde{V}_m(t^-)_x=\alpha\,\Delta v_x\,\chi(x\in\tilde{\partial}_2)\,dN(\tau,x,v)$ and

$$\mathbf{E}_m\left(\int_{s\wedge\tau_{B,\delta}}^{t\wedge\tau_{B,\delta}} du\,2\rho LM^{-1}\Phi_{1,x}\,V_m(u^-)_x\,2\rho_m\alpha Ld(u)\langle v_yF(\eta(X_m(u),v))\rangle\prod g_i\right)$$

$$=\mathbf{E}_m\Bigg(\iint_s^t dN(\tau,x,v)\alpha\,\Delta v_x\,\chi(x\in\tilde{\partial}_2)$$

$$\times\{2\rho_m\alpha Ld(\tau)\langle v_yF(\eta(X_m(\tau),v))\rangle\}\prod g_i\Bigg)$$

Then integration by parts and using (5.36) shows that

$$\mathbf{E}_m\left(\iint_s^t dN(\tau, x, v)\alpha\, \Delta v_x\, [G(\tau, t) - V_m(\tau^-)]_y\, \chi(x \in \tilde{\partial}_2) \prod g_i\right)$$

$$\approx \mathbf{E}_m\Big(2\rho_m \alpha L\{\tilde{V}_m(t^-)_x\, d(t)\langle v_y F(\eta(X_m(t), v))\rangle$$

$$- \tilde{V}_m(s^-)_x\, d(s)\langle v_y F(\eta(X_m(s), v))\rangle\} \prod g_i\Big)$$

$$- \mathbf{E}_m\Big(\int_s^t du\, 2\rho_m \alpha L \tilde{V}_m(u^-)_x\, \tilde{V}_m(u^-)_y$$

$$\times \langle F(\eta(X_m(u), v))\, \eta(X_m(u), v)\, F'(\eta(X_m(u), v))\rangle \prod g_i\Big) \tag{5.37}$$

Finally, we observe that $G(u', u)$ as a function of u is increasing and for all $\tau < s$, $\mathbf{E}_m(G(\tau, t) - G(\tau, s)) \leqslant D_4$ for some positive constant D_4; therefore,

$$\lim_{m \to 0} \mathbf{E}_m\left(\int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} dN(\tau, x, v)\alpha\, \Delta v_x\, \chi(x \in \tilde{\partial}_2)[G(\tau, t) - G(\tau, s)]_y \prod g_i\right) = 0 \tag{5.38}$$

so that for $f = f_3$ will easily follow from (5.34), (5.35), (5.37), and (5.38).

Finally, we shall consider the case $f = f_3$, i.e., we have to prove that

$$\lim_{m \to 0} \mathbf{E}_m\Big(\Big\{f_3(\tilde{V}_m(t)) - f_3(\tilde{V}_m(s)) - \int_{s \wedge \tau_{B,\delta}}^{t \wedge \tau_{B,\delta}} du\, (-2b_x \tilde{V}_m(u^-)_x\, \tilde{V}_m(u^-)_x)$$

$$- 2b_y(\tilde{X}_m(u), \tilde{V}_m(u^-))\, \tilde{V}_m(u^-)_x + [\sigma^2 + \sigma_{yy}^2(\tilde{X}_m(u))]\Big\} \prod g_i\Big) = 0 \tag{5.39}$$

Since by definition of the process $(G(t, \infty); t \geqslant 0)$ and (5.2) we have that

$$f_3(\tilde{V}_m(t)) - f_3(\tilde{V}_m(s))$$

$$= \int_s^t \int dN(\tau, x, v)(\alpha\, \Delta v_x)^2\, \chi(x \in \tilde{\partial}_2)$$

$$+ 2\int_s^t \int dN(\tau, x, v)\alpha\, \Delta v_x\, \tilde{V}_m(\tau^-)_x\, \chi(x \in \tilde{\partial}_2)$$

$$+ \int_s^t \int dN(\tau, x, v)\, W(\tau, x, v; \infty)_y^2\, \chi(x \in \tilde{\partial}_1, |v_y| > 2B)$$

$$+ 2\int_s^t \int dN(\tau, x, v)\, W(\tau, x, v; \infty)_y\, \chi(x \in \tilde{\partial}_1, |v_y| > 2B)$$

$$+ [\tilde{V}_m(t^-)_y^2 - G(t, \infty)_y^2] - [\tilde{V}_m(s^-)_y^2 - G(s, \infty)_y^2] \tag{5.40}$$

Then, in a similar fashion as in (5.34), one can easily see that

$$\lim_{m\to 0}\mathbf{E}_m\left(\left\{\int_s^t\int dN(\tau,x,v)(\alpha\,\Delta v_x)^2\,\chi(x\in\tilde{\partial}_2)\right.\right.$$
$$\left.\left.-\int_s^t\int dN(\tau,x,v)(\alpha\,\Delta v_x)^2\,\chi(x\in\tilde{\partial}_2)\right\}\prod g_i\right)$$
$$=\lim_{m\to 0}\mathbf{E}_m\left(\int_{s\,\wedge\,\tau_{B,\delta}}^{t\,\wedge\,\tau_{B,\delta}} du\,[\sigma^2-bV_m(u^-)_x]\prod g_i\right)\tag{5.41}$$

Next we shall deal with the expectation of the third term on the rhs of (5.40). Since $\alpha^2[\tilde{k}(u,\infty)+1]^2\,\Delta v_y^2$ is the only part of $W(\tau,x,v;\infty)_y^2$ that contributes in the limit $m\to 0$, by (5.23)–(5.28) and (5.30) with $\alpha^2 m^{-1/2}(v_y)^3$ instead of $\alpha m^{-1/2}\tilde{V}_m(u^-)_y$ one can prove that

$$\lim_{m\to 0}\mathbf{E}_m\left(\int_s^t\int dN(\tau,x,v)\;W(\tau,x,v;\infty)_y^2\,\chi(x\in\tilde{\partial}_1,\,|v_y|>2B)\prod g_i\right)$$
$$=\lim_{m\to 0}\mathbf{E}_m\left(\int_s^t du\,4\rho LM^{-2}\Phi_{3,y}\left[1+(\Phi_{3,y})^{-1}\iint_{|v_y|>2B}dv\,h_m(v)\,v_y^3\right.\right.$$
$$\left.\left.\times 2F(\eta_0(u,v))\right]\prod g_i\right)\tag{5.42}$$

Therefore, to complete the proof of (5.13) with $f=f_3$, we need only show that

$$\lim_{m\to 0}\mathbf{E}_m\left(\left\{2\int_s^t\int dN(\tau,x,v)\;W(\tau,x,v;\infty)_y\,G(\tau,\infty)_y\,\chi(x\in\tilde{\partial}_1,\,|v_y|>2B)\right.\right.$$
$$\left.\left.+[\tilde{V}_m(t^-)_y^2-G(t,\infty)_y^2]-[\tilde{V}_m(s^-)_y^2-G(s,\infty)_y^2]\right\}\prod g_i\right)$$
$$=\lim_{m\to 0}\mathbf{E}_m\left(-\int_s^t du\,2b_y(X_m(u),V_m(u^-))\,\tilde{V}_m(u^-)_y\prod g_i\right)\tag{5.43}$$

Since $G(u',u)\neq G(u',\infty)$ only if $u-\sigma_m<u'<u$ and since for all $T<+\infty$

$$\lim_{m\to 0}\mathbf{E}_m\left(\int_0^T\int dN(\tau,x,v)\,[W(\tau,x,v;\infty)_y^2-W(\tau,x,v;T)_y^2]\right.$$
$$\left.\times\chi(x\in\tilde{\partial}_1,\,|v_y|>2B)\right)=0$$

we have that the lhs of (5.43) is equal to

$$\lim_{m\to 0} \mathbf{E}_m \Bigg(\Big\{ 2 \int_s^t \int_{m\to 0} dN(\tau, x, v)\, W(\tau, x, v; \infty)_y\, G(\tau, \infty)_y\, \chi(x \in \tilde{\partial}_1, |v_y| > 2B)$$
$$- \int_{t-\sigma_m}^t \int dN(\tau, x, v)[W(\tau, x, v; \infty)_y - W(\tau, x, v; t)_y]\, G(\tau, t)_y$$
$$\times \chi(x \in \tilde{\partial}_1, |v_y| > 2B)$$
$$+ \int_{s-\sigma_m}^s \int dN(\tau, x, v)[W(\tau, x, v; \infty)_y - W(\tau, x, v; s)_y]\, G(\tau, s)_y$$
$$\times \chi(x \in \tilde{\partial}_1, |v_y| > 2B) \Big\} \prod g_i \Bigg) \tag{5.44}$$

Now note that

$$\mathbf{E}_m \Bigg(\int_{t-\sigma_m}^t \int dN(\tau, x, v)[W(\tau, x, v; \infty)_y - W(\tau, x, v; t)_y]\, G(\tau, t)_y$$
$$\times \chi(x \in \tilde{\partial}_1, |v_y| > 2B) \prod g_i \Bigg)$$
$$\approx \mathbf{E}_m \Bigg(\Big\{ \int_{t-\sigma_m}^t \int dN(\tau, x, v)[W(\tau, x, v; \infty)_y - W(\tau, x, v; t)_y]\, G(\tau, \infty)_y$$
$$\times \chi(x \in \tilde{\partial}_1, |v_y| > 2B)$$
$$- 1/2 \int_{t-\sigma_m}^t \int dN(\tau, x, v)[W(\tau, x, v; \infty)_y - W(\tau, x, v; t)_y]^2$$
$$\times \chi(x \in \tilde{\partial}_1, |v_y| > 2B) \Big\} \prod g_i \Bigg) \tag{5.45}$$

and set for all $u < t$

$$A_t(u) = \int_0^t du \int_{|v_y| > 2B} \lambda(u, dx, dv)[W(u, x, v; \infty)_y - W(u, x, v; t)_y]$$

and

$$M_t(u) = \int_0^t \int dN(\tau, x, v)[W(\tau, x, v; \infty)_y - W(\tau, x, v; t)_y]$$
$$\times\ \chi(x \in \tilde{\partial}_1, |v_y| > 2B) - A_t(u)$$

Then $(M_t(u); u \leqslant t)$ is a G_t-martingale and therefore

$$\mathbf{E}_m\left(\left\{\int_{t-\sigma_m}^{t}\int dN(\tau, x, v)[W(\tau, x, v; \infty)_y - W(\tau, x, v; t)_y]^2 \times \chi(x \in \tilde{\partial}_1, |v_y| > 2B)\right\} \prod g_i\right)$$

$$= \mathbf{E}_m\left([A_t(t)^2 + M_t(t)^2 - 2A_t(t)\, M_t(t)] \prod g_i\right)$$

$$\approx \mathbf{E}_m\left([4\rho L M^{-1}\langle F(\eta(X_m(t), v))\rangle]^2 \prod g_i\right) \tag{5.46}$$

To get the last equality we have also used (5.32), (4.38), and Schwarz's inequality to prove that

$$\lim_{m \to 0} \mathbf{E}_m\left([M_t(t)^2 - 2A_t(t)\, M_t(t)] \prod g_i\right) = 0$$

Thus, (5.43) will follow from (5.44)–(5.46) in a similar way as in (5.34)–(5.36).

APPENDIX

We shall prove

$$\mathbf{P}(\{\lim_{B \to \infty, \delta \to 0} \tau_{B,\delta} = \infty\}) = 1 \tag{A.1}$$

with $\tau_B = \inf\{t \geqslant 0: |V(t)| \geqslant B\}$ and $\tau_\delta = \inf\{t \geqslant 0: d(X(t)) < \delta\}$ we have $\tau_{B,\delta} = \tau_B \wedge \tau_\delta$. Since, for any $t < \tau_\delta$, $\mathrm{Tr}(\sigma(X(t)) \leqslant \mathrm{const} \cdot (1 + \delta^{-1})$ and $\langle b(X(t), V(t), V(t)\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbf{R}^2$, it follows (see, for example, ref. 7, Theorem 3, p. 33) that for any $T < +\infty$

$$\mathbf{P}(\{\lim_{B \to \infty} \tau_B < T \wedge \tau_\delta\}) = 0 \tag{A.2}$$

Therefore it remains to prove that

$$\mathbf{P}(\{\lim_{\delta \to 0} \tau_\delta = \infty\}) = 1 \tag{A.3}$$

Let $\bar{b}_y(x) = b_y(x, v)/v$. We shall prove (A.3) under the hypotheses that

$$\int_0^1 dx\, \bar{b}_y(x) = \infty$$

and

$$\sup_{x\geq 0} \bar{b}_y(x)/\sigma_{yy}^2(x) \geq \alpha \qquad \text{for some} \quad \alpha > 0$$

Let $(y(t), v(t); t>0)$ be the process defined by the y component of the process $(X(t), V(t); t\geq 0)$, i.e., $(y(t), v(t); t>0)$ is the process satisfying the following stochastic differential equations:

$$\begin{aligned} dy(t) &= v(t)\, dt \\ dv(t) &= -b_y(y(t), v(t))\, dt + \sigma_{yy}(y(t))\, dw(t) \end{aligned} \tag{A.4}$$

$y(0)=X(0)_y=y_0$, $v(0)=V(0)_y=v_0$, $(w(t); t\geq 0)$ standard Wiener process

Define $t=\int_0^{T(t)} ds\, \sigma_{yy}^2(y(s))$, and let $(z(t); t\geq 0)$ and $(u(t); t\geq 0)$ be the following processes:

$$\begin{aligned} u(t) &\equiv v(T(t)) \\ z(t) &= \int_{y_0}^{y(T(t))} dy\, \bar{b}_y(y) \\ &= \int_0^t ds\, v(T(s))\, \bar{b}_y(y(T(s)))[\sigma_{yy}^2(y(T(s)))]^{-1} \\ &\equiv B(y(T(t))) \end{aligned} \tag{A.5}$$

Then

$$\begin{aligned} du(t) &= -dz(t) + dw(t) \\ dz(t) &= u(t)\, \bar{b}_y(y(T(t)))[\sigma_{yy}^2(y(T(t)))]^{-1}\, dt \end{aligned} \tag{A.6}$$

For any $A>0$, let $(z_A(t); t\geq 0)\equiv(z(At)/\sqrt{A}; t\geq 0)$; then, $(z_A(t); t\geq 0)$ converges weakly as $A\to\infty$ to a Brownian motion $(\bar{w}(t); t\geq 0)$.

To prove this assertion, it is enough to show that for any $\varepsilon>0$

$$\lim_{A\to\infty} \mathbf{P}(\{\sup_{t\in[0,A]} |u(t)|/\sqrt{A} > \varepsilon\}) = 0 \tag{A.7}$$

For this note that from (A.6)

$$du(t) = -u(t)\, \bar{b}_y(y(T(s)))[\sigma_{yy}^2(y(T(s)))]^{-1}\, dt + dw(t)$$

and then observe that $\sup_{x\geq 0}\{\bar{b}_y(x)[\sigma_{yy}^2(x)]^{-1}\}\geq\alpha$; (A.7) follows by routine arguments.

Next we shall prove that

(i) $T(\infty)=\lim_{\delta\to 0}\tau_\delta\equiv\tau$

(ii) $T^{-1}\int_0^T ds\,\sigma_{yy}^2[B^{-1}(z(s))]^{-1}$

converges in distribution as $T\to\infty$ to

$$a^{-1}\int_0^1 ds\,\chi(\bar{w}(s)\in(0,\infty))$$

where $a=\lim_{x\to\infty}\sigma_{yy}^2(x)$. Note that

$$T(t)=\int_0^t ds\,\sigma_{yy}^2[B^{-1}(z(s))]^{-1}$$

and by (i)

$$\begin{aligned}\mathbf{P}(\{\lim_{\delta\to 0}\tau_\delta=\infty\})&=\mathbf{P}(\{T(\infty)=\infty\})\\&=\mathbf{P}\left(\left\{\int_0^\infty ds\,\sigma_{yy}^2[B^{-1}(z(s))]^{-1}=\infty\right\}\right)\end{aligned}\tag{A.8}$$

Therefore (A.3) will follow from (i), (ii), and (A.8), since by the arcsine law (see ref. 8)

$$\begin{aligned}&\lim_{\varepsilon\to 0}\limsup_{A\to\infty}\mathbf{P}\left(\left\{A^{-1}\int_0^A ds\,\sigma_{yy}^2[B^{-1}(z(s))]^{-1}<\varepsilon\right\}\right)\\&\quad=\lim_{\varepsilon\to 0}\mathbf{P}\left(\left\{\int_0^1 ds\,\chi(\bar{w}(s)\in(0,\infty))<a\varepsilon\right\}\right)=0\end{aligned}$$

Proof of (i). By (A.4′) it is sufficient to show that

$$\mathbf{P}\left(\left\{\int_0^\tau ds\,\sigma_{yy}^2(y(s))<\infty\right\}\right)=0\tag{A.9}$$

Let us assume that (A.9) does not hold; then there exists an $\alpha>0$ such that

$$\mathbf{P}\left(\left\{\int_0^\tau ds\,\sigma_{yy}^2[y(s)]^{-1}<\infty\right\}\right)>\alpha$$

and since $M(t)=\int_0^t dw(s)\, \sigma_{yy}(y(s))$ is a martingale with $\langle M \rangle_t = \int_0^t ds\, \sigma_{yy}^2(y(s))$, and $\lim_{t\to\tau} M(t)$ exists on a set of positive measure. By (A.4),

$$M(t)=V(t)+\int_{y_0}^{y(t)} dx\, \bar{b}_y(x)$$

and for all ω

$$\lim_{t\to\tau}\int_{y_0}^{y(t)} dx\, \bar{b}_y(x) = -\int_0^{y_0} dx\, \bar{b}_y(x) = -\infty$$

Hence $\lim_{t\to\tau} v(t)=+\infty$ on the set on which $\lim_{t\to\tau} M(t)$ exists. This implies that there exists an $\varepsilon>0$ such that $\int_{\tau-\varepsilon}^{\tau} ds\, v(s)>0$, but this is impossible since $\int_{\tau-\varepsilon}^{\tau} ds\, v(s) = -d(y(\tau-\varepsilon))<0$. Thus, (A.9) holds.

Proof of (ii). Note that $B^{-1}(0)=y_0$ and $\lim_{A\to\infty} B^{-1}(A^{1/2}z)=+\infty$ if $z>0$ and 0 if $z<0$; then we have that

$$\lim_{A\to\infty}\sigma_{yy}^2[B^{-1}(A^{1/2}z)]^{-1}=\begin{cases}\sigma_{yy}^2(y_0)^{-1} & \text{if} \quad z=0\\ \lim\limits_{x\to\infty}\sigma_{yy}^2(x)^{-1}=a & \text{if} \quad z>0\\ \lim\limits_{x\to 0+}\sigma_{yy}^2(x)^{-1}=0 & \text{if} \quad z<0\end{cases} \tag{A.10}$$

Thus, (ii) will easily follow from (A.10) and the weak convergence of $(z_A(t);\, t>0)$ to $(\bar{w}(t);\, t>0)$ by observing that

$$A^{-1}\int_0^A ds\, \sigma_{yy}^2[B^{-1}(z(s))]^{-1}=\int_0^1 ds\, \sigma_{yy}^2[B^{-1}(A^{1/2}z^A(s))]^{-1}$$

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